Theorem 1.4.3 (continued 2)

Proof (continued).

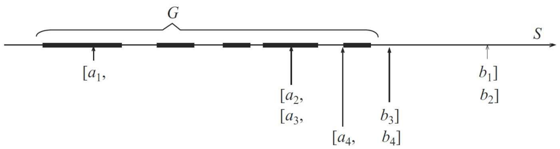

Figure 1.4.3. Proof by bisection: the first few intervals.

So sequence $\{b_i\}_{i=1}^{\infty}$ (of right had endpoints) is Cauchy, since for any positive k, for $m, n > N$ we have $b_m, b_n \in [a_N, b_N]$ and so $|b_m - b_n| \le b_N - a_N < k$. Since S is Cauchy complete by hypothesis, then $\{b_i\}$ converges and there is $c \in S$ such that $\lim_{i\to\infty} b_i = c$. Similarly, $\{a_i\}$ is Cauchy and for $m, n > N$ we have $a_m, a_n \in [a_N, b_N]$ and $|a_m - a_n| < k$. So $\{a_i\}$ converges as well; say $\lim_{i\to\infty} a_i = c'$. ASSUME $c' \neq c$. Then for $|c - c'|/3 = k$ we have $k > 0$ and there exists positive integers N_a and N_b such that for all $n > N_a$ we have $|a_n - c'| < |c - c'|/3$, and for all $n > N_b$ we have $|b_n - c| < |c - c'|/3$.

Theorem 1.4.3 (continued 3)

Theorem 1.4.3. An ordered field is order complete if and only if it is Cauchy complete and Archimedean.

Proof (continued). So for all $n > \max\{N_a, N_b\}$ we have we have

$$|c' - c| = |c' - a_n + a_n - b_n + b_n - c| \le |c' - a_n| + |a_n - b_n| + |b_n - c|$$
$$< |c - c'|/3 + |a_n - b_n| + |c - c'|/3$$

or

$$|a_n - b_n| > |c' - c|. \qquad (*)$$

But as shown above, for any positive k (such as $k = |c' - c|$ there is natural number N such that for all $n > N$ we have $|a_n - b_n| = b_n - a_n < k$. But this CONTRADICTS $(*)$, so the assumption that $c' \neq c$ is false, and hence $\lim_{i\to\infty} a_i = c = \lim_{i\to\infty} b_i$. Since $\{a_i\}$ is a monotone increasing sequence and $\{b_i\}$ is a monotone decreasing sequence, then for all $i \in \mathbb{N}$ we have $a_i \le c \le b_i$. So by the definition of limit, for any given positive integer k there is $a_n \in \{a_i\}$ and $b_m \in \{b_i\}$ such that

$$c - 1/k < a_n \le c \le b_m < c + 1/k. \qquad (**)$$

Theorem 1.4.3 (continued 4)

Proof (continued). ASSUME c is not an upper bound of G. Then there is $g \in G$ such that $g > c$. By the hypothesized Archimedean property of S, there is an integer k such that $1/k < g - c$. Then by $(**)$, $c \le b_m < c + 1/k < c + (g - c) = g$. But this CONTRADICTS the fact that (by construction) b_m is an upper bound of G. So the assumption that c is not an upper bound of G is false, and hence c is an upoer bound of G.

ASSUME c is not the least upper bound of G. Then there is an upper bound B with $B < c$. By the hypothesized Archimedean property of S, there is an integer k such that $1/k < c - B$. Then by $(**)$, $B < c - 1/k < a_n$, so that a_n is an upper bound of G. But this CONTRADICTS the construction of a_n where every a_n is NOT an upper bound of G (recall that $a_i = d_{i-1} = a_{i-1} + b_{i-1})/2$ only when d_i is not an upper bound of G). So the assumption that c is not the least upper bound of G is false, and hence x is the least upper bound of G. Since G is an arbitrary nonempty subset of S, then S is order complete, as claimed. □

Theorem 2.1.A

Theorem 2.1.A. The ordered field of real numbers $\mathbb{R}$ is Archimedean.

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be positive real numbers. Let $\{x(n)\}$ be in $\mathbf{x}$. Then $\{x(n)\}$ is positive and, by definition, there are natural numbers M and N so that for $n > N$ we have $x(n) > 1/M$. Define $\{x'(n)\} = \{x(N+1), x(N+2), \ldots\}$ (so that $\{x'(n)\}$ is a subsequence of $\{x(n)\}$). Then $\{x'(n)\}$ is also in $\mathbf{x}$ and is bounded below by the rational number $1/(2M)$. Let $\mathbf{a}$ be the equivalence class containing $\{a, a, \ldots\}$. Then $\mathbf{a} < \mathbf{x}$, where $\mathbf{a}$ is rational.

Let $\{y(n)\}$ be in $\mathbf{y}$. Now $\{y(n)\}$ is a Cauchy sequence of rational numbers, so for $\varepsilon = 1$ there is natural number $N(1)$ such that for all $m, n > N(1)$ we have $|y(n) - y(m)| < \varepsilon = 1$. Then $\{y(n)\}$ is bounded above by the rational number $\max\{y(1), y(2), \ldots, y(N(1)), y(N(1)+1)\}$. Let $b = \max\{y(1), y(2), \ldots, y(N(1)), y(N(1)+1)\} + 1$ and let $\mathbf{b}$ be the equivalence class containing $\{b, b, \ldots\}$. Then $\mathbf{y} < \mathbf{b}$, where $\mathbf{b}$ is rational.

Theorem 2.1.A (continued)

Theorem 2.1.A. The ordered field of real numbers $\mathbb{R}$ is Archimedean.

Proof (continued). So for any positive real numbers $\mathbf{x}$ and $\mathbf{y}$, we have rational $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{0} < \mathbf{a} < \mathbf{x}$ and $\mathbf{y} < \mathbf{b}$.

The rational numbers form an Archimedean field by Lemma 1.4.A, so for positive rational numbers a and b, we have that there is a natural number n such that $na > b$ (or $na - b$ is positive). For sequences $\{n, n, \ldots\}$, $\{a, a, \ldots\}$, and $\{b, b, \ldots\}$ we have $\{n, n, \ldots\} \cdot \{a, a, \ldots\} - \{b, b, \ldots\} = \{na - b, na - b, \ldots\}$ is positive. Therefore $\mathbf{na} - \mathbf{b}$ is positive, or $\mathbf{na} > \mathbf{b}$. Similarly $\mathbf{na} < \mathbf{nx}$, and so by transitivity of the ordering we have $\mathbf{y} < \mathbf{b} < \mathbf{na} < \mathbf{nx}$. That is, for any positive real numbers $\mathbf{x}$ and $\mathbf{y}$, there is a natural number $\mathbf{n}$ such that $\mathbf{nx} > \mathbf{y}$ so that $\mathbb{R}$ is Archimedean, as claimed. □

Theorem 2.1.7

Theorem 2.1.7. The real numbers $\mathbb{R}$ form an order complete ordered field.

Proof. We need to show that an arbitrary Cauchy sequence of real numbers converges to a real number. Let $\{\mathbf{x}(n)\}$ be a Cauchy sequence of real numbers. We want to show that this sequence converges to a real number $\mathbf{b}$. We do so by finding a Cauchy sequence of rational numbers $\{b_n\}_{n=1}^\infty$ and then consider the equivalence class $\mathbf{b}$ containing $\{b_n\}_{n=1}^\infty$.

For each $n \in \mathbb{N}$ let $\{x(n,i)\}_{i=1}^\infty$ be a representative of $\mathbf{x}(n)$. Then $\{x(n,i)\}_{i=1}^\infty$ is a Cauchy sequence of rational numbers, so there exists natural number N_n such that for all $j, k > N_n$ we have $|x(n,j) - x(n,k)| < 1/n$. Define $b_n = x(n, N_n + 1)$, so that

$$|x(n,i) - b_n| < 1/n \text{ for all } i > N_n. \qquad (1)$$

Then $\{b_n\}$ is a sequence of rational numbers.

Theorem 2.1.7 (continued 1)

Theorem 2.1.7. The real numbers $\mathbb{R}$ form an order complete ordered field.

Proof (continued). Next, we show that $\{b_n\}_{n=1}^\infty$ is Cauchy. Let $\varepsilon > 0$ where ε is rational (we consider a rational ε since, by definition, this is what is needed to show a sequence of rationals is Cauchy). Since ordered field $\mathbb{Q}$ is Archimedean by Lemma 1.4.A, there is natural number $N^* = N^*(\varepsilon)$ such that

$$1/N^* < \varepsilon/3 \qquad (2)$$

Since $\{\mathbf{x}(n)\}_{n=1}^\infty$ is a Cauchy sequence of real numbers by hypothesis, then with $\boldsymbol{\varepsilon}/\mathbf{3} = (\varepsilon/3, \varepsilon/3, \ldots)$ we have by Note RU.K that there exists natural number $N^{**} = N^{**}(\varepsilon)$ such that for all $m, n > N^{**}$ there are natural numbers M_c and N_c (dependent on m and n) where for all $i > N_c$ we have $\varepsilon/3 - |x(n,i) - x(m,i)| > 1/M_c$ or

$$|x(n,i) - x(m,i)| < \varepsilon/3 - 1/M_c < \varepsilon/3. \qquad (3)$$

Theorem 2.1.7 (continued 2)

Theorem 2.1.7. The real numbers $\mathbb{R}$ form an order complete ordered field.

Proof (continued). Let $N = \max\{N^*, N^{**}\}$ and suppose $m, n > N$. (Notice that N depends on ε and not on m and/or n.) Since $m, n > N^{**}$ then there exist natural numbers M' and N' such that for all $i > N'$ we have by (3) that

$$|x(n,i) - x(m,i)| < \varepsilon/3 - 1/M' < \varepsilon/3. \qquad (4)$$

For $i > N_n$ we have by (1) that

$$|b_n - x(n,i)| < 1/n \qquad (5)$$

and for $i > N_m$ we have by (1) that

$$|b_m - x(m,i)| < 1/m. \qquad (6)$$

Theorem 2.1.7 (continued 3)

Proof (continued). So for any $i > \max\{N', N_n, N_m\}$ (notice the value of i depends on m and n) we have

$$\begin{aligned} |b_n - b_m| &= |b_n - x(n,i) + x(n,i) - x(m,i) + x(m,i) - b_m| \\ &\leq |b_n - x(n,i)| + |x(n,i) - x(m,i)| + |x(m,i) - b_m| \\ &\quad \text{by the Triangle Inequality in } \mathbb{Q} \\ &< 1/n + \varepsilon/3 + 1/m \text{ by (5) (since } i > N_n), \\ &\quad \text{(4) (since } m,n > N^{**} \text{ and } i > N'), \text{ and} \\ &\quad \text{(6) (since } i > N_m), \text{ respectively)} \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \text{ by (2) since } m,n > N^*. \end{aligned}$$

That is, for all $m, n > N$ we have $|b_n - b_m| < \varepsilon$. Therefore, $\{b_n\}_{n=1}^\infty$ is a Cauchy sequence of rational numbers. Hence, the equivalence class containing $\{b_n\}_{n=1}^\infty$, $\mathbf{b}$, is a real number.

Theorem 2.1.7 (continued 4)

Theorem 2.1.7. The real numbers $\mathbb{R}$ form an order complete ordered field.

Proof (continued). To prove $\mathbf{b}$ is the limit of $\{\mathbf{x}(n)\}_{n=1}^\infty$, let $\varepsilon > \mathbf{0}$ where ε is a real number (we consider a real ε since this is needed to show convergence of a sequence of real numbers). Let $\{e(i)\}_{i=1}^\infty$ be a representative of ε. Since ε is positive, then by the definition of a positive Cauchy sequence of rational numbers. So (by definition) there are natural numbers M' and N' so that for all $i > N'$ we have $e(i) > 1/M'$. Consider the constant sequence $\{1/M'\}_{i=1}^\infty$ as a representative of real number $\mathbf{1/M'}$. We now have $\mathbf{1/M'} < \varepsilon$.

As above, let $\{x(n,i)\}_{i=1}^\infty$ be a representative of $\mathbf{x}(n)$. Let $N(\varepsilon) = 2M'$. Then for all $n > N(\varepsilon)$ we have by (1) that there is a natural number N_n such that if $i > N_n$ then $|x(n,i) - b_n| < 1/n < 1/(2M')$.

Theorem 2.1.7 (continued 5)

Theorem 2.1.7. The real numbers $\mathbb{R}$ form an order complete ordered field.

Proof (continued). That is, for all $n > N(\varepsilon)$ there are natural numbers M and N, namely $N = N_n$ and $M = 2M'$, such that if $i > N$ then $|x(n,i) - b_n| < 1/(2M') = 1/M' - 1/(2M')$ or $1/M' = |x(n,i) - b_n| > 1/(2M')$ and hence $e(i) = |x(n,i) - b_n| > 1/M' = |x(n,i) - b_n| > 1/(2M') = 1/M$. By Note RU.L, this gives (in terms of real numbers) $|\mathbf{x}(n) - \mathbf{b}| < \mathbf{1/M'} < \varepsilon$. Therefore $\mathbf{x}(n)$ converges to $\mathbf{b}$.

Since $\mathbf{x}(n)$ is an arbitrary Cauchy sequence of real numbers, then every Cauchy sequence of real numbers converges. That is, $\mathbb{R}$ is Cauchy complete. By Theorem 1.2.A, $\mathbb{R}$ is Archimedean, so by Theorem 1.4.3 we have that $\mathbb{R}$ is order complete, as claimed. □

Theorem 1.3.1

Theorem 1.3.1. The function $i : \mathbb{N} \cup \{0\} \to S$, where S is an ordered field, satisfies:

(a) $i(n+m) = i(n) + i(m)$ for all $m, n \in \mathbb{N} \cup \{0\}$,

(b) $i(nm) = i(n)i(m)$ for all $m, n \in \mathbb{N} \cup \{0\}$, and

(c) i is one to one on $\mathbb{N} \cup \{0\}$.

Proof. (a) This holds trivially for $n = 0$. We give an inductive proof on n. Let $m \in \mathbb{N}$ be arbitrary but fixed. For the base case $n = 1$, we have

$$\begin{aligned} i(1+m) &= \underbrace{1_S + 1_S + \cdots + 1_S + 1_S}_{1+m \text{ times}} = 1_S + (\underbrace{1_S + 1_S + \cdots + 1_S}_{m \text{ times}}) \\ &= i(1) + i(m) \text{ by the definition of } i. \end{aligned}$$

For the induction hypothesis, suppose for $n = k \geq 1$ we have $i(k+m) = i(k) + i(m)$.

Theorem 1.3.1 (continued 1)

Theorem 1.3.1. The function $i : \mathbb{N} \cup \{0\} \to S$, where S is an ordered field, satisfies:

(a) $i(n+m) = i(n) + i(m)$ for all $m, n \in \mathbb{N} \cup \{0\}$.

Proof (continued). Now consider:

$$\begin{aligned} i((k+1)+m) &= i((k+m)+1) = i(k+m_{+}1_S \text{ by the base case,} \\ &\quad \text{where } m \text{ is replaced with } k+m \\ &= (i(k)+i(m)) + 1_S \text{ by the induction hypothesis} \\ &= (i(k)+1_S) + i(m) = i(k+1) + i(m) \text{ by the} \\ &\quad \text{base case where } m \text{ is replaced with } k+1. \end{aligned}$$

So the result holds for $n = k+1$ giving the induction step. Therefore, the claim holds for all $n \in \mathbb{N}$ by mathematical induction and, since $m \in \mathbb{N}$ is arbitrary, $i(n+m) = i(n) + i(m)$ for all $m, n \in \mathbb{N}$, as claimed.

Theorem 1.3.1 (continued 2)

Theorem 1.3.1. The function $i : \mathbb{N} \cup \{0\} \to S$, where S is an ordered field, satisfies:

(b) $i(nm) = i(n)i(m)$ for all $m, n \in \mathbb{N} \cup \{0\}$.

Proof (continued). (b) Again, we give a inductive proof on n. Let $m \in \mathbb{N}$ be arbitrary but fixed. For the base case $n = 1$, we have $i(1m) = i(m) = 1_S i(m) = i(1)i(m)$. For the inductive hypothesis, suppose for $n = k \geq 1$ we have $i(km) = i(k)i(m)$. Now consider:

$$\begin{aligned} i((k+1)m) &= i(km+m) = i(km) + i(m) \text{ by part (a)} \\ &= i(k)i(m) + i(m) \text{ by the induction hypothesis} \\ &= i(k+1)i(m(\text{ by part (a).} \end{aligned}$$

So the result holds for $n = k+1$, giving the induction step . Therefore, the claim holds for all $n \in \mathbb{N}$ by mathematical induction and, since $m \in \mathbb{N}$ is arbitrary, $i(nm) = i(n)i(m)$ for all $m, n \in \mathbb{N}$, as claimed.

Theorem 1.3.1 (continued 3)

Theorem 1.3.1. The function $i : \mathbb{N} \cup \{0\} \to S$, where S is an ordered field, satisfies:

(c) i is one to one on $\mathbb{N} \cup \{0\}$.

Proof (continued). (c) To show i is one to one, suppose $i(m) = i(n)$ for some $m, n \in \mathbb{N}$ where, WLOG, say $n \geq m$. Then

$$i(n) = i(n-m+m) = i(n-m) + i(m) \text{ or } i(n) - i(m) = i(n-m)$$

or (since $i(n) = i(m)$) $0_S = i(n-m)$. But the only nonnegative integer mapped mapped to 0_S is 0, so that $n - m = 0$ and $n = m$. That is, i is one to one on $\mathbb{N} \cup \{0\}$, as claimed. □

Theorem 2.3.3

Theorem 2.3.B. (Problem 2.3.1) Mapping $i : \mathbb{Q} \to S$ is well-defined. That is, $i(p/q) = i(r/s)$ for $p/q = r/s$ where $p, q, r, s \in \mathbb{Z}$.

Proof. Notice that rational numbers p/q and r/s (where $p, q, r, s \in \mathbb{Z}$ with $q \neq 0$ and $s \neq 0$) are equal if and only if $ps = qr$. If $r = 0$ then $ps = qr = q(0) = 0$ so that $p = 0$ since $s \neq 0$. Then $p/q = r/s = 0$ and $i(p/q) = i(r/s) = 0_S$. So we assume WLOG that $r \neq 0$. With $p/q = r/s$ we have

$$\begin{aligned} i(p/q) &= r((p/q)(1_S)) = i((p/q)(r/s)(s/r)) \text{ since } r \neq 0 \\ &= i((ps)/(qr) \cdot (r/s)) = i(1_S \cdot (r/s)) \text{ since } ps = qr \\ &= i(r/s). \end{aligned}$$

Hence, the value of i on an element of $\mathbb{Q}$ is independent of the representative of the element. That is, i is well-defined on $\mathbb{Q}$, as claimed. □

Theorem 2.3.1

Theorem 2.3.1. The function $i : \mathbb{Q} \to S$ is a field and order isomorphism from the $\mathbb{Q}$ onto a subfield of S.

Proof. The image of i is $S_{\mathbb{Q}} = \{i(p/q) \mid p/q \in \mathbb{Q}\}$. Of course i is onto its image, so i maps $\mathbb{Q}$ onto $S_{\mathbb{Q}}$. Consider $i(p_1/q_1), i(p_2/q_2) \in S_{\mathbb{Q}}$ where $i(p_1/q_1) = i(p_2/q_2)$. Then $i(p_1)(i(q_1))^{-1} = i(p_2)(i(q_2))^{-1}$ or $i(p_1)i(q_2) = i(p_2)i(q_1)$ or $i(p_1q_2) = i(p_2q_1)$ by Theorem 1.3.1(b) and Theorem 1.3.2(b). Since i is one to one on $\mathbb{Z}$ by Theorem 1.3.2(c), then $p_1q_2 = p_2q_1$, or $p_1/q_1 = p_2/q_2$ so that i is on to one on $\mathbb{Q}$.

Let $p_1/q_1, p_2/q_2 \in \mathbb{Q}$. Then

$$\begin{aligned} i\left(\frac{p_1}{q_1}\cdot\frac{p_2}{q_2}\right) &= i\left(\frac{p_1p_2}{q_1q_2}\right) = i(p_1p_2)/i(q_1q_2) \text{ by the definition of } i \\ &= i(p_1)i(p_2)/(i(q_1)i(q_2)) \text{ by Theorems 1.3.1(b) and 1.3.2(b)} \\ &= \frac{i(p_1)}{i(q_1)}\cdot\frac{i(p_2)}{i(q_2)} = i\left(\frac{p_1}{q_1}\right) i\left(\frac{p_2}{q_2}\right) \text{ by the definition of } i. \end{aligned}$$

Theorem 2.3.1 (continued 1)

Proof (continued). Also,

$$\begin{aligned} i\left(\frac{p_1}{q_1}+\frac{p_2}{q_2}\right) &= i\left(\frac{p_1q_2+p_2q_1}{q_1q_2}\right) = \frac{i(p_1q_2+p_2q_1)}{i(q_1q_2)} \text{ by the definition of } i \\ &= (i(p_1q_2)+i(p_2q_1))/(i(q_1)i(q_2)) \text{ by Theorems 1.3.1(a)} \\ &\quad \text{and 1.3.2(a)} \\ &= \frac{i(p_1)i(q_2)+i(p_2)i(q_1)}{i(q_1)i(q_2)} \text{ by Theorems 1.3.1(b)} \\ &\quad \text{and 1.3.2(b)} \\ &= \frac{i(p_1)i(q_2)}{i(q_1)i(q_2)} + \frac{i(p_2)i(q_1)}{i(q_1)i(q_2)} = \frac{i(p_1)}{i(q_1} + \frac{i(p_2)}{i(q_2)} \\ &= i(p_1/q_1) + i(p_2/q_2) \text{ by the definition of } i. \end{aligned}$$

Therefore $i : \mathbb{Q} \to S_{\mathbb{Q}}$ is a ring isomorphism. Since $\mathbb{Q}$ is a field, then $S_{\mathbb{Q}}$ is a subfield of S and i is actually a field isomorphism.

Theorem 2.3.1 (continued 2)

Proof (continued). Now to show that i is an order isomorphism; that is, i preserves the order. As observed in Note RU.N, i maps positive integers to positive elements of S and maps negative integers to negative elements of S. Notice that for positive n in an ordered field we also have that $n^{-1} = 1/n$ by Kirkwood's Exercise 1.2.7(b) is positive. So for any positive $p/q \in \mathbb{Q}$ (say both p and q are positive integers) we have $i(p/q) = i(p)(i(q))^{-1}$ where $i(p)$ and $i(q)$ are positive. Since S is an ordered field and $i(q)$ is positive, then Kirkwood's Exercise 1.2.7(b) gives that $(i(q))^{-1}$ is positive. So by the closure of the positive set in S, $i(p)(i(q))^{-1} = i(p)/i(q) = i(p/q)$ is positive. Hence i maps all positive rationals to positive elements of S. Similarly, if $p/q \in \mathbb{Q}$ is negative (say $p < 0$ and $q > 0$) we have $i(p)$ is negative in S (as observed above), so $i(p/q) = i(p)(i(q))^{-1}$ is negative in S by Kirkwood's Theorem 1-7(d) (which implies that a positive times a negative is negative). So the positive set in S corresponds exactly to the positive set in $\mathbb{Q}$ under i. That is, i is a field and order isomorphism, as claimed. □