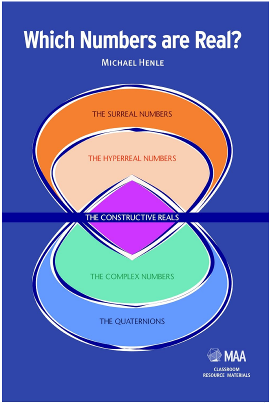
Analysis 1

Supplement. The Real Numbers are the Unique Complete Ordered Field



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Theorem 2.1.1(a)

Theorem 2.1.1. Let $x = x\{x(n)\}$ and $y = \{y(n)\}$ be sequences of rational numbers.

(1) If x and y are Cauchy sequences, then so are $\{x(n) + y(n)\}$ and $\{x(n)y(n)\}$.

Proof. (1) Let $\varepsilon > 0$. Since $x(n)$ is a Cauchy sequence, then there is natural number $N_x(\varepsilon)$ such that for all $m, n > N_x(\varepsilon)$ we have $|x(n) - x(m)| < \varepsilon/2$. Since $y(n)$ is a Cauchy sequence, then there is natural number $N_y(\varepsilon)$ such that for all $m, n > N_y(\varepsilon)$ we have $|y(n) - y(m)| < \varepsilon/2$. Let $N(\varepsilon) = \max\{N_x(\varepsilon), N_y(\varepsilon)\}$.

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$$\begin{aligned} & |(x(n) + y(n) - (x(m) + y(m))| \\ = & \ |(x(n) - x(m)) + (y(n) - y(m))| \\ \leq & \ |x(n) - x(m)| + |y(n) - y(m)| \text{ by the Triangle Inequality} \\ < & \ \varepsilon/2 + \varepsilon/2 = \varepsilon \text{ since } m, n > N_x(\varepsilon) \text{ and } m, n > N_y(\varepsilon). \end{aligned}$$

So $\{x(n) + y(n)\}$ is Cauchy, as claimed.

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$$\begin{aligned}
&= |(x(n) - x(m)) + (y(n) - y(m))| \\
&\le |x(n) - x(m)| + |y(n) - y(m)| \text{ by the Triangle Inequality} \\
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\end{aligned}$$

So $\{x(n) + y(n)\}$ is Cauchy, as claimed.

Theorem 2.1.1(a) (continued 1)

Theorem 2.1.1. Let $x = x\{x(n)\}$ and $y = \{y(n)\}$ be sequences of rational numbers.

(1) If x and y are Cauchy sequences, then so are $\{x(n) + y(n)\}$ and $\{x(n)y(n)\}$.

Proof (continued). We saw that a Cauchy sequence of real numbers is bounded in Kirkwood's Exercise 2.3.13(a). In Exercises 1.4.7 and 1.4.11, we have that a Cauchy sequence of rational numbers is bounded above and below by integers. Since $\{x(n)\}$ is a Cauchy sequence of real numbers, then there is natural number B_x such that $|x(n)| \leq B_x$ for all $n \in \mathbb{N}$. Similarly, since $\{y(n)\}$ is a Cauchy sequence of real numbers, then there is natural number B_y such that $|y(n)| \leq B_y$ for all $n \in \mathbb{N}$. Then $B = \max\{B_x, B_y\}$ is a bound for $|x(n)|$ and $|y(n)|$ for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$. Since $x(n)$ is a Cauchy sequence, then there is natural number $N_x'(\varepsilon)$ such that for all $m, n > N_x'(\varepsilon)$ we have $|x(n) - x(m)| < \varepsilon/(2B)$.

Theorem 2.1.1(a) (continued 1)

Theorem 2.1.1. Let $x = x\{x(n)\}$ and $y = \{y(n)\}$ be sequences of rational numbers.

(1) If x and y are Cauchy sequences, then so are $\{x(n) + y(n)\}$ and $\{x(n)y(n)\}$.

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Theorem 2.1.1(a) (continued 2)

Proof (continued). Since $y(n)$ is a Cauchy sequence, then there is natural number $N'_y(\varepsilon)$ such that for all $m, n > N'_y(\varepsilon)$ we have $|y(n) - y(m)| < \varepsilon/(2B)$. Let $N'(\varepsilon) = \max\{N'_x(\varepsilon), N'_y(\varepsilon)\}$. For $m, n > N'(\varepsilon)$ we have

$$\begin{aligned}
& |(x(n)y(n) - x(m)y(m)| \\
= \;& |x(n)y(n) - x(n)y(m) + x(n)y(m) - x(m)y(m)| \\
\leq \;& |x(n)y(n) - x(n)y(m)| + |x(n)y(m) - x(m)y(m)| \\
& \text{ by the Triangle Inequality} \\
= \;& |x(n)||y(n) - y(m)| + |x(n) - x(m)||y(m)| \\
\leq \;& B|y(n) - y(m) + |x(n) - x(m)|B \\
< \;& (B)\varepsilon/(2B) + \varepsilon/(2B) \times (B) = \varepsilon.
\end{aligned}$$

So $\{x(n)y(n)\}$ is Cauchy, as claimed. $\square$

Theorem 2.1.3

Theorem 2.1.3. If $\{x(n)\}$ and $\{x'(n)\}$ are equivalent Cauchy sequences of rational numbers, and likewise for $\{y(n)\}$ and $\{y'(n)\}$, then $\{x(n) + y(n)\}$ and $\{x'(n) + y'(n)\}$ are equivalent, and $\{x(n)y(n)\}$ and $\{x'(n)y'(n)\}$ are equivalent.

Proof. First, since $\{x(n)\}$ and $\{y(n)\}$ are Cauchy sequences, then so are $\{x(n) + y(n)\}$ and $\{x(n)y(n)\}$ by Theorem 2.1.1(a). Let $\{x(n)\}$ and $\{x'(n)\}$ be equivalent Cauchy sequences, and likewise for $\{y(n)\}$ and $\{y'(n)\}$. Then $\{x(n) - x'(n)\}$ and $\{y(n) - y'(n)\}$ are null sequences.

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By Theorem 2.1.1(b),

$$\{(x(n)+y(n))-(x'(n)+y'(n)\} = \{(x(n)-x'(n))+(y(n)-y'(n))\}$$

$$= \{x(n)-x'(n)\}+\{y(n)-y'(n)\}$$

is a null sequence. That is, $\{x(n)+y(n)\} \sim \{x'(n)+y'(n)\}$, as claimed.

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By Theorem 2.1.1(b),

$$\{(x(n) + y(n)) - (x'(n) + y'(n)\} = \{(x(n) - x'(n)) + (y(n) - y'(n))\}$$

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is a null sequence. That is, $\{x(n) + y(n)\} \sim \{x'(n) + y'(n)\}$, as claimed.

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Proof. Next

$$\begin{aligned}\{x(n)y(n)-x'(n)y'(n))\} &= \{x(n)y(n)-x(n)y'(n))+x(n)y'(n)-x'(n)y'(n)\} \\ &= \{x(n)(y(n)-y'(n))+y'(n)(x(n)-x'(n))\} \\ &= \{x(n)(y(n)-y'(n))\}+\{y'(n)(x(n)-x'(n))\}.\end{aligned}$$

Since sequences $\{x(n)-x'(n)\}$ and $\{y(n)-y'(n)\}$ are null sequences then by Theorem 2.1.1(3), sequences $\{x(n)(y(n)-y'(n))\}$ and $\{y'(n)(x(n)-x'(n))\}$ are also null sequences. By Theorem 2.1.1(2), $\{x(n)(y(n)-y'(n))\}+\{y'(n)(x(n)-x'(n))\}$ is a null sequence, so that $\{x(n)y(n)-x'(n)y'(n))\}$ is a null sequence. Hence $\{x(n)y(n)\} \sim \{x'(n)y'(n)\}$, as claimed. □

Theorem 2.1.4

Theorem 2.1.4. $\mathbb{R}$ is a field.

Proof. We need to verify the eight parts (or "axioms") of the definition of "field" given in Section 1.2. Properties of the Real Numbers as an Ordered Field.

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A1. We claim that $+$ and $\cdot$ are binary operations. We just need to confirm that, for given $\mathbf{x}$ and $\mathbf{y}$, the sum and product of these are uniquely determined. By definition, $+$ and $\cdot$ each take a pair of equivalence classes of Cauchy sequences of rational numbers (that is, a pair of real numbers) and produce an equivalence class of Cauchy sequences of rational numbers (i.e., real number). By Theorem 2.1.3, the product and sum are uniquely determined. That is, $+$ and $\cdot$ are binary operations, as claimed.

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A2. We claim that $+$ and $\cdot$ are associative. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be real numbers. Suppose $\{x(n)\} \in \mathbf{x}$, $\{y(n)\} \in \mathbf{y}$, and $\{z(n)\} \in \mathbf{z}$.

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Theorem 2.1.4 (continued 1)

Theorem 2.1.4. $\mathbb{R}$ is a field.

Proof (continued). Then $\mathbf{x} + (\mathbf{y} + \mathbf{z})$ is the equivalence class containing

$$\begin{aligned} \{x(n)\} + (\{y(n)\} + \{z(n)\}) &= \{x(n) + (y(n) + z(n))\} \\ &= \{(x(n) + y(n)) + z(n)\} \text{ since } + \text{ is} \\ &\quad \text{ associative in } \mathbb{Q} \\ &= (\{x(n)\} + \{y(n)\}) + \{z(n)\}. \end{aligned}$$

Since $(\{x(n)\} + \{y(n)\}) + \{z(n)\}$ is in the equivalence class $(\mathbf{x} + \mathbf{y}) + \mathbf{z}$, then we have $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$, as claimed. Similarly, $\mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z})$ is the equivalence class containing

$$\begin{aligned} \{x(n)\} \cdot (\{y(n)\} \cdot \{z(n)\}) &= \{x(n) \cdot (y(n) \cdot z(n))\} \\ &= \{(x(n) \cdot y(n)) \cdot z(n)\} \text{ since } \cdot \text{ is} \\ &\quad \text{ associative in } \mathbb{Q} \\ &= (\{x(n)\} \cdot \{y(n)\}) \cdot \{z(n)\}. \end{aligned}$$

Theorem 2.1.4 (continued 1)

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Proof (continued). Then $\mathbf{x} + (\mathbf{y} + \mathbf{z})$ is the equivalence class containing

$$\begin{aligned} \{x(n)\} + (\{y(n)\} + \{z(n)\}) &= \{x(n) + (y(n) + z(n))\} \\ &= \{(x(n) + y(n)) + z(n)\} \text{ since } + \text{ is associative in } \mathbb{Q} \\ &= (\{x(n)\} + \{y(n)\}) + \{z(n)\}. \end{aligned}$$

Since $(\{x(n)\} + \{y(n)\}) + \{z(n)\}$ is in the equivalence class $(\mathbf{x} + \mathbf{y}) + \mathbf{z}$, then we have $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$, as claimed. Similarly, $\mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z})$ is the equivalence class containing

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Theorem 2.1.4 (continued 2)

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Proof (continued). Since $(\{x(n)\} \cdot \{y(n)\}) \cdot \{z(n)\}$ is in the equivalence class $(\mathbf{x} \cdot \mathbf{y}) \cdot \mathbf{z}$, then we have $\mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) \cdot \mathbf{z}$, as claimed.

A3. We claim that $+$ and $\cdot$ are commutative. Let $\mathbf{x}$ and $\mathbf{y}$ be real numbers. Suppose $\{x(n)\} \in \mathbf{x}$ and $\{y(n)\} \in \mathbf{y}$. Then $\mathbf{x} + \mathbf{y}$ is the equivalence class containing

$$\begin{aligned} \{x(n)\} + \{y(n)\} &= \{x(n) + y(n)\} \\ &= \{y(n) + x(n)\} \text{ since } + \text{ is commutative in } \mathbb{Q} \\ &= \{\{y(n)\} + \{x(n)\}. \end{aligned}$$

Since $\{y(n)\} + \{x(n)\}$ is in the equivalence class $\mathbf{y} + \mathbf{x}$, then we have $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.

Theorem 2.1.4 (continued 2)

Theorem 2.1.4. $\mathbb{R}$ is a field.

Proof (continued). Since $(\{x(n)\} \cdot \{y(n)\}) \cdot \{z(n)\}$ is in the equivalence class $(\mathbf{x} \cdot \mathbf{y}) \cdot \mathbf{z}$, then we have $\mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) \cdot \mathbf{z}$, as claimed.

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Since $\{y(n)\} + \{x(n)\}$ is in the equivalence class $\mathbf{y} + \mathbf{x}$, then we have $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.

Theorem 2.1.4 (continued 3)

Proof (continued). Similarly, $\mathbf{x} \cdot \mathbf{y}$ is the equivalence class containing

$$\begin{aligned} \{x(n)\} \cdot \{y(n)\} &= \{x(n) \cdot y(n)\} \\ &= \{y(n) \cdot x(n)\} \text{ since } \cdot \text{ is commutative in } \mathbb{Q} \\ &= \{\{y(n)\} \cdot \{x(n)\}. \end{aligned}$$

Since $\{y(n)\} \cdot \{x(n)\}$ is in the equivalence class $\mathbf{y} \cdot \mathbf{x}$, then we have $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$, as claimed.

A4. We claim that $\cdot$ distributes over $+$. With the notation above, we have

$$\begin{aligned} \{x(n)\}(\{y(n)\} + \{z)n)\}) &= \{x(n)(y(n) + z(n)\} \\ &= \{x(n)y(n) + x(n)z(n)\} \text{ since } \cdot \text{ distributes} \\ &\quad \text{over } + \text{ in } \mathbb{Q} \\ &= \{x(n)\}\{y(n)\} + \{x(n)\}\{z(n)\}. \end{aligned}$$

So the equivalence class $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z})$ and the equivalence class $\mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$ are equal. That is, $\cdot$ distributes over $+$ as claimed.

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Proof (continued). Similarly, $\mathbf{x} \cdot \mathbf{y}$ is the equivalence class containing

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So the equivalence class $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z})$ and the equivalence class $\mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$ are equal. That is, $\cdot$ distributes over $+$ as claimed.

Theorem 2.1.4 (continued 4)

Proof (continued). A5. We claim there exist additive and multiplicative identities. Denote the equivalence class containing the Cauchy sequence of rationals $\{x(n)\}_{n=1}^{\infty}$ where $x(n) = 0$ for all $n \in \mathbb{N}$ as $\mathbf{0}$ (notice then that $\mathbf{0}$ is the equivalence class of all null sequences). We then have

$$\begin{aligned} \{x(n)\} + \{y(n)\} &= \{0 + y(n)\} \\ &= \{y(n)\} \text{ since 0 is the additive identity in } \mathbb{Q} \end{aligned}$$

So for any real number $\mathbf{y}$, the equivalence class $\mathbf{0} + \mathbf{y}$ and the equivalence class $\mathbf{y}$ are equal. That is, $\mathbf{0}$ is the additive identity in $\mathbb{R}$.

Similarly with $\{x(n)\}_{n=1}^{\infty}$, where $x(n) = 1$ for all $n \in \mathbb{N}$, denoted as $\mathbf{1}$ we have

$$\begin{aligned} \{x(n)\} \cdot \{y(n)\} &= \{1 \cdot y(n)\} \\ &= \{y(n)\} \text{ since 1 is the multiplicative identity in } \mathbb{Q} \end{aligned}$$

So for any real number $\mathbf{y}$, the equivalence class $\mathbf{1} \cdot \mathbf{y}$ and the equivalence class $\mathbf{y}$ are equal. That is, $\mathbf{1}$ is the multiplicative identity in $\mathbb{R}$.

Theorem 2.1.4 (continued 4)

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So for any real number $\mathbf{y}$, the equivalence class $\mathbf{0} + \mathbf{y}$ and the equivalence class $\mathbf{y}$ are equal. That is, $\mathbf{0}$ is the additive identity in $\mathbb{R}$.

Similarly with $\{x(n)\}_{n=1}^{\infty}$, where $x(n) = 1$ for all $n \in \mathbb{N}$, denoted as $\mathbf{1}$ we have

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Theorem 2.1.4 (continued 5)

Proof (continued). A6. We claim that every real number has an additive inverse. Let $\{x(n)\}$ be any Cauchy sequence of rationals. Define sequence $\{y(n)\}_{n=1}^{\infty}$ where $y(n) = -x(n)$ for all $n \in \mathbb{N}$. We then have

$$\begin{aligned} \{x(n)\} + \{y(n)\} &= \{x(n) + y(n)\} = \{x(n) - x(n)\} \\ &= \{0\}_{n=1}^{\infty} \text{ since } \mathbb{Q} \text{ has additive inverses.} \end{aligned}$$

So the equivalence class containing $\{x(n)\} + \{y(n)\}$ is the same as the equivalence class containing $\{0\}_{n=1}^{\infty}$. Denoting the equivalence class containing $\{y(n)\} = \{-x(n)\}$ as $-\mathbf{x}$, we have $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$ (or simply "$\mathbf{f} - \mathbf{f} = \mathbf{0}$"). So every real number has an additive inverse, as claimed.

A7. We claim that every non-additive-identity (i.e., "nonzero") has a multiplicative inverse. The additive identity $\mathbf{0}$ is the equivalence class of all null Cauchy sequences of rational numbers, so consider $\{x(n)\}$ a non-null Cauchy sequences of rational numbers.

Theorem 2.1.4 (continued 5)

Proof (continued). A6. We claim that every real number has an additive inverse. Let $\{x(n)\}$ be any Cauchy sequence of rationals. Define sequence $\{y(n)\}_{n=1}^{\infty}$ where $y(n) = -x(n)$ for all $n \in \mathbb{N}$. We then have

$$\begin{aligned} \{x(n)\} + \{y(n)\} &= \{x(n) + y(n)\} = \{x(n) - x(n)\} \\ &= \{0\}_{n=1}^{\infty} \text{ since } \mathbb{Q} \text{ has additive inverses.} \end{aligned}$$

So the equivalence class containing $\{x(n)\} + \{y(n)\}$ is the same as the equivalence class containing $\{0\}_{n=1}^{\infty}$. Denoting the equivalence class containing $\{y(n)\} = \{-x(n)\}$ as $-\mathbf{x}$, we have $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$ (or simply "$\mathbf{f} - \mathbf{f} = \mathbf{0}$"). So every real number has an additive inverse, as claimed.

A7. We claim that every non-additive-identity (i.e., "nonzero") has a multiplicative inverse. The additive identity $\mathbf{0}$ is the equivalence class of all null Cauchy sequences of rational numbers, so consider $\{x(n)\}$ a non-null Cauchy sequences of rational numbers.

Theorem 2.1.4 (continued 6)

Proof (continued). Then by Henle's Exercise 2.1.7, we have that there are natural numbers M and N such that for all $n > N$ we have $|x(n)| > 1/M$. For such M and N, define $\{y(n)\}_{n=1}^{\infty}$ where

$$y(n) = \begin{cases} x(n) & \text{if } n \leq N \\ 1/x(n) & \text{if } x > N. \end{cases}$$

We now have that

$$x(n)y(n) = \begin{cases} x(n)^2 & \text{if } n \leq N \\ 1 & \text{if } x > N. \end{cases}$$

So $\{x(n)\} \cdot \{y(n)\} = \{x(n)y(n)\}$ converges to 1 and hence is in the equivalence class $\mathbf{1}$ of part A5 above. That is, the equivalence class containing $\mathbf{x} \cdot \mathbf{y}$ is the same as the equivalence class $\mathbf{1}$ (i.e., the multiplicative identity). Denoting $\mathbf{y}$ as $\mathbf{x}^{-1}$, we have $\mathbf{x} \cdot \mathbf{x}^{-1} = \mathbf{1}$, and so every nonzero real number has a multiplicative inverse, as claimed. □

Theorem 2.1.4 (continued 6)

Proof (continued). Then by Henle's Exercise 2.1.7, we have that there are natural numbers M and N such that for all $n > N$ we have $|x(n)| > 1/M$. For such M and N, define $\{y(n)\}_{n=1}^{\infty}$ where

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Theorem 2.1.5

Theorem 2.1.5. Let $\mathbf{x} \in \mathbb{R}$. If one sequence from equivalence class $\mathbf{x}$ is positive, then all sequences in $\mathbf{x}$ are positive.

Proof. Suppose $\{x(n)\} \in \mathbf{x}$ is positive. Then, by definition of "positive," there are positive natural numbers M_1 and N_1 such that for all $n > N_1$ we have $x(n) > 1/M_1$. Let $\{y(n)\} \in \mathbf{x}$. Then $\{x(n)\} \sim \{y(n)\}$ and so $\{x(n) - y(n)\}$ is a null sequence (by definition of $\sim$), or $\{x(n) - y(n)\} \to 0$. This means that for all rational numbers $\varepsilon > 0$, there is natural number $N_2 = N(\varepsilon)$ such that for all $n > N_2$ we have $|x(n) - y(n)| < \varepsilon$.

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$$1/(2M_1) > |x(n) - y(n)| \geq x(n) - y(n) > 1/M_1 - y_n$$

$$\text{or } y_n > 1/M_1 - 1/(2M_1) = 1/(2M_1).$$

Theorem 2.1.5

Theorem 2.1.5. Let $\mathbf{x} \in \mathbb{R}$. If one sequence from equivalence class $\mathbf{x}$ is positive, then all sequences in $\mathbf{x}$ are positive.

Proof. Suppose $\{x(n)\} \in \mathbf{x}$ is positive. Then, by definition of "positive," there are positive natural numbers M_1 and N_1 such that for all $n > N_1$ we have $x(n) > 1/M_1$. Let $\{y(n)\} \in \mathbf{x}$. Then $\{x(n)\} \sim \{y(n)\}$ and so $\{x(n) - y(n)\}$ is a null sequence (by definition of $\sim$), or $\{x(n) - y(n)\} \to 0$. This means that for all rational numbers $\varepsilon > 0$, there is natural number $N_2 = N(\varepsilon)$ such that for all $n > N_2$ we have $|x(n) - y(n)| < \varepsilon$. In particular, for $\varepsilon = 1/(2M_1)$ there is natural number N_3 such that for all $n > N_3$ we have $|x(n) - y(n)| < 1/(2M_1)$. With $N = \max\{N_1, N_3\}$ we now have for all $n > N$ that both $x(n) > 1/M_1$ and $|x(n) - y(n)| < 1/(2M_1)$, and hence

$$1/(2M_1) > |x(n) - y(n)| \geq x(n) - y(n) > 1/M_1 - y_n$$

$$\text{or } y_n > 1/M_1 - 1/(2M_1) = 1/(2M_1).$$

Theorem 2.1.5 (continued)

Theorem 2.1.5. Let $\mathbf{x} \in \mathbb{R}$. If one sequence from equivalence class $\mathbf{x}$ is positive, then all sequences in $\mathbf{x}$ are positive.

Proof (continued). So we have natural numbers M and N (namely, $M = 2M_1$ and $N = \max\{N_1, N_3\}$ where M_1, N_1, and N_3 are as above) such that for all $n > N$ we have $y(n) > 1/M$. That is, $\{y(n)\}$ is a positive sequence. Since $\{y(n)\}$ is an arbitrary sequence of $\mathbf{x}$, then all sequences in $\mathbf{x}$ are positive, as claimed. □

Theorem 2.1.6

Theorem 2.1.6. The field of real numbers $\mathbb{R}$ is ordered.

Proof. We know that $\mathbb{R}$ is a field by Theorem 2.1.4, so we just need to verify that the set of positive equivalence classes is (i) closed under addition, (ii) closed under multiplication, and (iii) satisfies the Law of Trichotomy (see "Axiom 8/Definition of Ordered Field" in Section 1.2. Properties of the Real Numbers as an Ordered Field).

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First, suppose $\mathbf{x}$ and $\mathbf{y}$ are both positive. Let $\{x(n)\}$ be a Cauchy sequence of rationals in $\mathbf{x}$, and let $\{y(n)\}$ be a Cauchy sequence of rationals in $\mathbf{y}$. Then by definition there are natural numbers M_x and N_x such that for all $n > N_x$ we have $x(n) > 1/M_x$, and there are natural numbers M_y and N_y such that for all $n > N_y$ we have $y(n) > 1/M_y$. Define positive integers $M = \lceil M_x M_y/(M_x + M_y) \rceil$ and $N = \max\{N_x, N_y\}$. Then for all $n > N$ we have

$$x(n) + y(n) > \frac{1}{M_x} + \frac{1}{M_y} = \frac{M_x + M_y}{M_x M_y} = \left(\frac{M_x + M_y}{M_x M_y} \right)^{-1} \geq \frac{1}{M}.$$

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First, suppose $\mathbf{x}$ and $\mathbf{y}$ are both positive. Let $\{x(n)\}$ be a Cauchy sequence of rationals in $\mathbf{x}$, and let $\{y(n)\}$ be a Cauchy sequence of rationals in $\mathbf{y}$. Then by definition there are natural numbers M_x and N_x such that for all $n > N_x$ we have $x(n) > 1/M_x$, and there are natural numbers M_y and N_y such that for all $n > N_y$ we have $y(n) > 1/M_y$. Define positive integers $M = \lceil M_x M_y/(M_x + M_y)\rceil$ and $N = \max\{N_x, N_y\}$. Then for all $n > N$ we have

$$x(n) + y(n) > \frac{1}{M_x} + \frac{1}{M_y} = \frac{M_x + M_y}{M_x M_y} = \left(\frac{M_x + M_y}{M_x M_y}\right)^{-1} \geq \frac{1}{M}.$$

Theorem 2.1.6 (continued 1)

Theorem 2.1.6. The field of real numbers $\mathbb{R}$ is ordered.

Proof (continued). Therefore, by definition, $\{x(n)\} + \{y(n)\} = \{x(n) + y(n)\}$ is positive, so that $\mathbf{x} + \mathbf{y}$ is positive. That is, the set of positive equivalence classes is (i) closed under addition, as claimed.

With $\mathbf{x}$, $\mathbf{y}$, $\{x(n)\}$, $\{y(n)\}$, and $M_x, M_y, N_x, N_y \in \mathbb{N}$ as above, define positive integers $M = \lceil M_x M_y \rceil$ and $N = \max\{N_x, N_y\}$. Then for all $n > N$ we have

$$x(n)y(n) > \frac{1}{M_x}\frac{1}{M_y} = \frac{1}{M_x M_y} = (M_x M_y)^{-1} \geq \frac{1}{M}.$$

Therefore, by definition, $\{x(n)\} \cdot \{y(n)\} = \{x(n)y(n)\}$ is positive, so that $\mathbf{x} \cdot \mathbf{y}$ is positive. That is, the set of positive equivalence classes is (ii) closed under multiplication, as claimed.

Theorem 2.1.6 (continued 1)

Theorem 2.1.6. The field of real numbers $\mathbb{R}$ is ordered.

Proof (continued). Therefore, by definition, $\{x(n)\} + \{y(n)\} = \{x(n) + y(n)\}$ is positive, so that $\mathbf{x} + \mathbf{y}$ is positive. That is, the set of positive equivalence classes is (i) closed under addition, as claimed.

With $\mathbf{x}$, $\mathbf{y}$, $\{x(n)\}$, $\{y(n)\}$, and $M_x, M_y, N_x, N_y \in \mathbb{N}$ as above, define positive integers $M = \lceil M_x M_y \rceil$ and $N = \max\{N_x, N_y\}$. Then for all $n > N$ we have

$$x(n)y(n) > \frac{1}{M_x}\frac{1}{M_y} = \frac{1}{M_x M_y} = (M_x M_y)^{-1} \geq \frac{1}{M}.$$

Therefore, by definition, $\{x(n)\} \cdot \{y(n)\} = \{x(n)y(n)\}$ is positive, so that $\mathbf{x} \cdot \mathbf{y}$ is positive. That is, the set of positive equivalence classes is (ii) closed under multiplication, as claimed.

Theorem 2.1.6 (continued 2)

Theorem 2.1.6. The field of real numbers $\mathbb{R}$ is ordered.

Proof (continued). Now for the Law of Trichotomy. First we show that the sets $\{\mathbf{x} \in \mathbb{R} \mid \mathbf{x} \text{ is positive}\}$, $\{\mathbf{x} \in \mathbb{R} \mid -\mathbf{x} \text{ is positive}\}$, and $\{\mathbf{x} \in \mathbb{R} \mid \mathbf{x} \text{ is null}\}$ are disjoint.

Suppose $\mathbf{x}$ is positive. Then for some $\{x(n)\}$ in $\mathbf{x}$, there are natural numbers M and N such that for all $n > N$ we have $x(n) > 1/M$. Then we cannot have that $\{x(n)\}$ is null, as we see by considering positive $\varepsilon = 1/M$. Also, for $\{-x(n)\}$ we know that for all $n > N$ we have $-x(n) < -1/M$ so that we cannot have $\{-x(n)\}$ positive (since all terms $-x(n)$ are less than 0 whenever $n > N$ and so cannot be greater than $1/M$ for any positive integer M); that is, $\{-x(n)\}$ is not positive. Therefore, there are no positive real numbers $\mathbf{x}$ for which $-\mathbf{x}$ is also positive or $\mathbf{x} = \mathbf{0}$.

Theorem 2.1.6 (continued 2)

Theorem 2.1.6. The field of real numbers $\mathbb{R}$ is ordered.

Proof (continued). Now for the Law of Trichotomy. First we show that the sets $\{\mathbf{x} \in \mathbb{R} \mid \mathbf{x} \text{ is positive}\}$, $\{\mathbf{x} \in \mathbb{R} \mid -\mathbf{x} \text{ is positive}\}$, and $\{\mathbf{x} \in \mathbb{R} \mid \mathbf{x} \text{ is null}\}$ are disjoint.

Suppose $\mathbf{x}$ is positive. Then for some $\{x(n)\}$ in $\mathbf{x}$, there are natural numbers M and N such that for all $n > N$ we have $x(n) > 1/M$. Then we cannot have that $\{x(n)\}$ is null, as we see by considering positive $\varepsilon = 1/M$. Also, for $\{-x(n)\}$ we know that for all $n > N$ we have $-x(n) < -1/M$ so that we cannot have $\{-x(n)\}$ positive (since all terms $-x(n)$ are less than 0 whenever $n > N$ and so cannot be greater than $1/M$ for any positive integer M); that is, $\{-x(n)\}$ is not positive. Therefore, there are no positive real numbers $\mathbf{x}$ for which $-\mathbf{x}$ is also positive or $\mathbf{x} = \mathbf{0}$.

Theorem 2.1.6 (continued 3)

Theorem 2.1.6. The field of real numbers $\mathbb{R}$ is ordered.

Proof (continued). Suppose $-\mathbf{x}$ is positive. Then for some $\{x(n)\}$ in $\mathbf{x}$, there are natural numbers M and N such that for all $n > N$ we have $-x(n) > 1/M$. Then we cannot have that $\{-x(n)\}$ is null, as we see by considering positive $\varepsilon = 1/M$. Also, for $\{-(-x(n))\} = \{x(n)\}$ we know that for all $n > N$ we have $x(n) < -1/M$ so that we cannot have $\{x(n)\}$ positive (since all terms $x(n)$ are less than 0 whenever $n > N$ and so cannot be greater than $1/M$ for any natural number M); that is, $\{x(n)\}$ is not positive. Therefore, there are no positive real numbers $-\mathbf{x}$ for which $\mathbf{x}$ is also positive or $-\mathbf{x} = \mathbf{0}$.

Suppose $\mathbf{x}$ is null (i.e., $\mathbf{x} = \mathbf{0}$). Let $\{x(n)\}$ be in $\mathbf{x} = \mathbf{0}$. Then for all rational $\varepsilon > 0$ there is integer $N(\varepsilon)$ such that for all $n > N(\varepsilon)$ we have $|x(n) - 0| = |x(n)| < \varepsilon$. For any natural number M, with $\varepsilon = 1/M$ we know that there is natural number $N(1/M)$ such that for all $n > N(1/M)$ we have $|x(n)| < 1/M$.

Theorem 2.1.6 (continued 3)

Theorem 2.1.6. The field of real numbers $\mathbb{R}$ is ordered.

Proof (continued). Suppose $-\mathbf{x}$ is positive. Then for some $\{x(n)\}$ in $\mathbf{x}$, there are natural numbers M and N such that for all $n > N$ we have $-x(n) > 1/M$. Then we cannot have that $\{-x(n)\}$ is null, as we see by considering positive $\varepsilon = 1/M$. Also, for $\{-(-x(n))\} = \{x(n)\}$ we know that for all $n > N$ we have $x(n) < -1/M$ so that we cannot have $\{x(n)\}$ positive (since all terms $x(n)$ are less than 0 whenever $n > N$ and so cannot be greater than $1/M$ for any natural number M); that is, $\{x(n)\}$ is not positive. Therefore, there are no positive real numbers $-\mathbf{x}$ for which $\mathbf{x}$ is also positive or $-\mathbf{x} = \mathbf{0}$.

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Theorem 2.1.6 (continued 4)

Theorem 2.1.6. The field of real numbers $\mathbb{R}$ is ordered.

Proof (continued). Hence, there cannot be a natural number N such that for all $n > N$ we have either $x(n) > 1/M$ or $-x(n) > 1/M$. That is, $\{x(n)\}$ is not positive and $\{-x(n)\}$ is not positive. Hence, the real number $\mathbf{0}$ is not positive and the real number $-\mathbf{0}$ is not positive. Therefore, the sets $\{\mathbf{x} \in \mathbb{R} \mid \mathbf{x} \text{ is positive}\}$, $\{\mathbf{x} \in \mathbb{R} \mid -\mathbf{x} \text{ is positive}\}$, and $\{\mathbf{x} \in \mathbb{R} \mid \mathbf{x} \text{ is null}\}$ are disjoint, as claimed. (We have given extra details here by considering all cases twice.)

Finally, we need to show that for every real number $\mathbf{x}$, either $\mathbf{x}$ is positive, $-\mathbf{x}$ is positive, or $\mathbf{x} = \mathbf{0}$. To prove this, suppose real number $\mathbf{f}$ is not positive and $-\mathbf{f}$ is not positive. Let $\{x(n)\}$ be in $\mathbf{f}$.

Theorem 2.1.6 (continued 4)

Theorem 2.1.6. The field of real numbers $\mathbb{R}$ is ordered.

Proof (continued). Hence, there cannot be a natural number N such that for all $n > N$ we have either $x(n) > 1/M$ or $-x(n) > 1/M$. That is, $\{x(n)\}$ is not positive and $\{-x(n)\}$ is not positive. Hence, the real number $\mathbf{0}$ is not positive and the real number $-\mathbf{0}$ is not positive. Therefore, the sets $\{\mathbf{x} \in \mathbb{R} \mid \mathbf{x} \text{ is positive}\}$, $\{\mathbf{x} \in \mathbb{R} \mid -\mathbf{x} \text{ is positive}\}$, and $\{\mathbf{x} \in \mathbb{R} \mid \mathbf{x} \text{ is null}\}$ are disjoint, as claimed. (We have given extra details here by considering all cases twice.)

Finally, we need to show that for every real number $\mathbf{x}$, either $\mathbf{x}$ is positive, $-\mathbf{x}$ is positive, or $\mathbf{x} = \mathbf{0}$. To prove this, suppose real number $\mathbf{f}$ is not positive and $-\mathbf{f}$ is not positive. Let $\{x(n)\}$ be in $\mathbf{f}$. Let $\varepsilon = p/q > 0$ be rational. Since $\{x(n)\}$ is not positive, then there are infinitely many $x(n)$ satisfying $x(n) \leq 1/(2q)$ (or else we could take natural number $M = 2q$ and there would then be a final $x(n)$ with $x(n) \leq 1/(2q)$ so that no natural number N would exist with $n > N$ implying $x(n) > 1/M$).

Theorem 2.1.6 (continued 4)

Theorem 2.1.6. The field of real numbers $\mathbb{R}$ is ordered.

Proof (continued). Hence, there cannot be a natural number N such that for all $n > N$ we have either $x(n) > 1/M$ or $-x(n) > 1/M$. That is, $\{x(n)\}$ is not positive and $\{-x(n)\}$ is not positive. Hence, the real number $\mathbf{0}$ is not positive and the real number $-\mathbf{0}$ is not positive. Therefore, the sets $\{\mathbf{x} \in \mathbb{R} \mid \mathbf{x} \text{ is positive}\}$, $\{\mathbf{x} \in \mathbb{R} \mid -\mathbf{x} \text{ is positive}\}$, and $\{\mathbf{x} \in \mathbb{R} \mid \mathbf{x} \text{ is null}\}$ are disjoint, as claimed. (We have given extra details here by considering all cases twice.)

Finally, we need to show that for every real number $\mathbf{x}$, either $\mathbf{x}$ is positive, $-\mathbf{x}$ is positive, or $\mathbf{x} = \mathbf{0}$. To prove this, suppose real number $\mathbf{f}$ is not positive and $-\mathbf{f}$ is not positive. Let $\{x(n)\}$ be in $\mathbf{f}$. Let $\varepsilon = p/q > 0$ be rational. Since $\{x(n)\}$ is not positive, then there are infinitely many $x(n)$ satisfying $x(n) \le 1/(2q)$ (or else we could take natural number $M = 2q$ and there would then be a final $x(n)$ with $x(n) \le 1/(2q)$ so that no natural number N would exist with $n > N$ implying $x(n) > 1/M$).

Theorem 2.1.6 (continued 5)

Theorem 2.1.6. The field of real numbers $\mathbb{R}$ is ordered.

Proof (continued). Similarly, since $\{-x(n)\}$ is not positive, then there are infinitely many $x(n)$ satisfying $-x(n) \leq 1/(2q)$ (i.e., $x(n) \geq 1/(2q)$). Since $\{x(n)\}$ is Cauchy, there exists natural number $N(1/(2q))$ such that for all $m, n > N(1/(2q))$ we have $|x(n) - x(m)| < 1/(2q)$. Now one such $x(m)$ (infinitely many, in fact) is less than or equal to $1/(2q)$, so for all $n > N(1/(2q))$ we must have $x(n) < 1/(2q) + 1/(2q) = 1/q \leq \varepsilon$. Also, one such $x(m)$ is greater than or equal to $-1/(2q)$, so for all $n > N(1/(2q))$ we must have $x(n) > -1/(2q) - 1/(2q) = -1/q \geq -\varepsilon$. That is, for all $n > N(1/2q)$ we have $|x(n) - 0| = |x(n)| < \varepsilon$. Hence, $\{x(n)\} \to 0$, $\{x(n)\}$ is a null sequence, and $\mathbf{x} = \mathbf{0}$. Therefore, for every real number $\mathbf{x}$, either $\mathbf{x}$ is positive, $-\mathbf{x}$ is positive (that is, $\mathbf{x}$ is negative), or $\mathbf{x}$ is null (that is, $\mathbf{x} = \mathbf{0}$). This verifies that the set of real numbers $\{\mathbf{x} \in \mathbb{R} \mid \mathbf{x} \text{ is positive}\}$ satisfies the definition of "positive set" in an ordering of a field. □

Theorem 2.1.6 (continued 5)

Theorem 2.1.6. The field of real numbers $\mathbb{R}$ is ordered.

Proof (continued). Similarly, since $\{-x(n)\}$ is not positive, then there are infinitely many $x(n)$ satisfying $-x(n) \le 1/(2q)$ (i.e., $x(n) \ge 1/(2q)$). Since $\{x(n)\}$ is Cauchy, there exists natural number $N(1/(2q))$ such that for all $m, n > N(1/(2q))$ we have $|x(n) - x(m)| < 1/(2q)$. Now one such $x(m)$ (infinitely many, in fact) is less than or equal to $1/(2q)$, so for all $n > N(1/(2q))$ we must have $x(n) < 1/(2q) + 1/(2q) = 1/q \le \varepsilon$. Also, one such $x(m)$ is greater than or equal to $-1/(2q)$, so for all $n > N(1/(2q))$ we must have $x(n) > -1/(2q) - 1/(2q) = -1/q \ge -\varepsilon$. That is, for all $n > N(1/2q)$ we have $|x(n) - 0| = |x(n)| < \varepsilon$. Hence, $\{x(n)\} \to 0$, $\{x(n)\}$ is a null sequence, and $\mathbf{x} = \mathbf{0}$. Therefore, for every real number $\mathbf{x}$, either $\mathbf{x}$ is positive, $-\mathbf{x}$ is positive (that is, $\mathbf{x}$ is negative), or $\mathbf{x}$ is null (that is, $\mathbf{x} = \mathbf{0}$). This verifies that the set of real numbers $\{\mathbf{x} \in \mathbb{R} \mid \mathbf{x} \text{ is positive}\}$ satisfies the definition of "positive set" in an ordering of a field. $\square$

Lemma 1.4.A

Lemma 1.4.A. (**Exercise 1.4.14** in Henle.) The ordered field of rational numbers $\mathbb{Q}$ is Archimedean.

Proof. Let $a = p_1/q_1$ and $b = p_2/q_2$ be positive rational numbers where p_1, q_1, p_2, q_2 are (positive) natural numbers. If $b < a$ we take $n = 1$ so that $b < 1a = a$. If $a \le b$ then $p_1/q_1 \le p_2/q_2$ and by Kirkwood's Theorem 1-7(c) this implies $p_1 q_2 \le p_2 q_1$. Since $1 \le p_1 q_2$ and $0 < p_2/q_2$ then $(1)(p_2 q_1) \le (p_1 q_2)(p_2 q_1)$, also by Kirkwood's Theorem 1-7(c). By Kirkwood's Example 1.2.7(b) we have that $1/q_1$ and $1/q_2$ are positive.

Lemma 1.4.A

Lemma 1.4.A. (**Exercise 1.4.14** in Henle.) The ordered field of rational numbers $\mathbb{Q}$ is Archimedean.

Proof. Let $a = p_1/q_1$ and $b = p_2/q_2$ be positive rational numbers where p_1, q_1, p_2, q_2 are (positive) natural numbers. If $b < a$ we take $n = 1$ so that $b < 1a = a$. If $a \leq b$ then $p_1/q_1 \leq p_2/q_2$ and by Kirkwood's Theorem 1-7(c) this implies $p_1 q_2 \leq p_2 q_1$. Since $1 \leq p_1 q_2$ and $0 < p_2/q_2$ then $(1)(p_2 q_1) \leq (p_1 q_2)(p_2 q_1)$, also by Kirkwood's Theorem 1-7(c). By Kirkwood's Example 1.2.7(b) we have that $1/q_1$ and $1/q_2$ are positive. Therefore $(p_2 q_1)/(q_1 q_2) \leq (p_1 q_2)(p_2 q_1)/(q_1 q_2)$ or $p_2/q_2 \leq (p_2 q_1)(p_1/q_1)$. Now $0 < 1$ (Kirkwood's Exercise 1.2.7(a)), so by Kirkwood's Theorem 1-7(a), $0 + (p_2 q_1) < 1 + (p_2 q_1)$, or by Theorem 1-7(c), $(p_2 q_1)(p_1/q_1) < (1 + p_2 q_1)(p_1/q_1)$. So with $n = 1 + p_2 q_1$ we have $b = p_2/q_2 < (1 + p_2 q_1)(p_1/q_1) = na$. Therefore $\mathbb{Q}$ is Archimedean, as claimed. □

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Theorem 1.4.3

Theorem 1.4.3. An ordered field is order complete if and only if it is Cauchy complete and Archimedean.

Proof. Let S denote the ordered field. First, suppose S is order complete. Then by Theorem 1.4.2 (or Kirkwood's Exercises 2.3.13 and 2.2.14), S is Cauchy complete. By Kirkwood's Theorem 1-18 (The Archimedean Principle), S is Archimdean. Hence, if S is order complete then it is Cauchy complete and Archimedean (as we establish in Analysis 1 [MATH 4217/5217] in Chapter 1, The Real Number System, and Chapter 2, Sequences of Real Numbers).

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Now suppose S is Cauchy complete and Archimedean. Let G be a nonempty subset of S that is bounded above. Let $g \in G$ and define $a_1 = g - 1$. Let b_1 be an upper bound of G strictly greater than a_1. Consider interval $[a_1, b_1]$ and let $d_1 = (a_1 + b_1)/2$ (the midpoint of the interval; notice 1 is the multiplicative identity of S and "2" denotes $1 + 1$, and any positive integer k is similarly represented in S).

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Theorem 1.4.3 (continued 1)

Theorem 1.4.3. An ordered field is order complete if and only if it is Cauchy complete and Archimedean.

Proof (continued). Either d is an upper bound for G or it is not. If d is an upper bound, set $s_1 = a_1$ and $b_2 = d$; if d is not an upper bound, set $a_2 = d$ and $b_2 = b_1$. Consider interval $[a_2, b_2]$ and notice that the length of $[a_2, b_2]$ is half the length of $[a_1, b_1]$. Let $d_2 = (a_2 + b_2)/2$, and iterate the process to create $[a_3, b_3]$, $[a_4, b_4]$, etc. At each stage, we have b_i is an upper bound of G, a_i is not an upper bound of G, and interval $[a_i, b_i]$ has length $(b_1 - a_i)/2^i$. Let $k \in S^+$ (that is, k is a positive element of ordered field S). Since S is Archimedean by hypothesis, there is integer N (notice that $N = \underbrace{1 + 1 + \cdots + 1}_{N \text{ times}}$) such that $(b_1 - a_1)/N < k$. Now $N \leq 2^N$ for all positive integers $N \in S$ (as is easily shown by mathematical induction), so the length of Nth interval $[a_N, b_N]$ is

$$b_N = a_N = (b_1 - a_1)/2^N \leq (b_1 - a_1)/N < k.$$

Theorem 1.4.3 (continued 1)

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Theorem 1.4.3 (continued 2)

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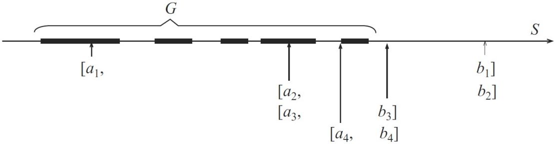

Figure 1.4.3. Proof by bisection: the first few intervals.

So sequence $\{b_i\}_{i=1}^{\infty}$ (of right had endpoints) is Cauchy, since for any positive k, for $m, n > N$ we have $b_m, b_n \in [a_N, b_N]$ and so $|b_m - b_n| \leq b_N - a_N < k$. Since S is Cauchy complete by hypothesis, then $\{b_i\}$ converges and there is $c \in S$ such that $\lim_{i\to\infty} b_i = c$. Similarly, $\{a_i\}$ is Cauchy and for $m, n > N$ we have $a_m, a_n \in [a_N, b_N]$ and $|a_m - a_n| < k$. So $\{a_i\}$ converges as well; say $\lim_{i\to\infty} a_i = c'$. ASSUME $c' \neq c$. Then for $|c - c'|/3 = k$ we have $k > 0$ and there exists positive integers N_a and N_b such that for all $n > N_a$ we have $|a_n - c'| < |c - c'|/3$, and for all $n > N_b$ we have $|b_n - c| < |c - c'|/3$.

Theorem 1.4.3 (continued 2)

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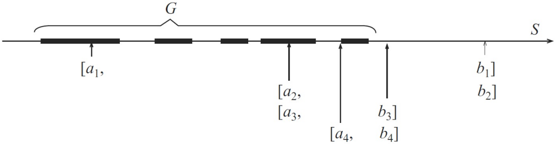

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Theorem 1.4.3 (continued 3)

Theorem 1.4.3. An ordered field is order complete if and only if it is Cauchy complete and Archimedean.

Proof (continued). So for all $n > \max\{N_a, N_b\}$ we have we have

$$|c' - c| = |c' - a_n + a_n - b_n + b_n - c| \le |c' - a_n| + |a_n - b_n| + |b_n - c|$$
$$< |c - c'|/3 + |a_n - b_n| + |c - c'|/3$$

or
$$|a_n - b_n| > |c' - c|. \qquad (*)$$

But as shown above, for any positive k (such as $k = |c' - c|$ there is natural number N such that for all $n > N$ we have $|a_n - b_n| = b_n - a_n < k$. But this CONTRADICTS $(*)$, so the assumption that $c' \neq c$ is false, and hence $\lim_{i\to\infty} a_i = c = \lim_{i\to\infty} b_i$. Since $\{a_i\}$ is a monotone increasing sequence and $\{b_i\}$ is a monotone decreasing sequence, then for all $i \in \mathbb{N}$ we have $a_i \le c \le b_i$. So by the definition of limit, for any given positive integer k there is $a_n \in \{a_i\}$ and $b_m \in \{b_i\}$ such that

$$c - 1/k < a_n \le c \le b_m < c + 1/k. \qquad (**)$$

Theorem 1.4.3 (continued 3)

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Theorem 1.4.3 (continued 4)

Proof (continued). ASSUME c is not an upper bound of G. Then there is $g \in G$ such that $g > c$. By the hypothesized Archimedean property of S, there is an integer k such that $1/k < g - c$. Then by $(**)$, $c \le b_m < c + 1/k < c + (g - c) = g$. But this CONTRADICTS the fact that (by construction) b_m is an upper bound of G. So the assumption that c is not an upper bound of G is false, and hence c is an upoer bound of G.

ASSUME c is not the least upper bound of G. Then there is an upper bound B with $B < c$. By the hypothesized Archimedean property of S, there is an integer k such that $1/k < c - B$. Then by $(**)$, $B < c - 1/k < a_n$, so that a_n is an upper bound of G. But this CONTRADICTS the construction of a_n where every a_n is NOT an upper bound of G (recall that $a_i = d_{i-1} = a_{i-1} + b_{i-1})/2$ only when d_i is not an upper bound of G). So the assumption that c is not the least upper bound of G is false, and hence x is the least upper bound of G. Since G is an arbitrary nonempty subset of S, then S is order complete, as claimed. $\square$

Theorem 1.4.3 (continued 4)

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Theorem 2.1.A

Theorem 2.1.A. The ordered field of real numbers $\mathbb{R}$ is Archimedean.

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be positive real numbers. Let $\{x(n)\}$ be in $\mathbf{x}$. Then $\{x(n)\}$ is positive and, by definition, there are natural numbers M and N so that for $n > N$ we have $x(n) > 1/M$. Define $\{x'(n)\} = \{x(N+1), x(N+2), \ldots\}$ (so that $\{x'(n)\}$ is a subsequence of $\{x(n)\}$). Then $\{x'(n)\}$ is also in $\mathbf{x}$ and is bounded below by the rational number $1/(2M)$. Let $\mathbf{a}$ be the equivalence class containing $\{a, a, \ldots\}$. Then $\mathbf{a} < \mathbf{x}$, where $\mathbf{a}$ is rational.

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Let $\{y(n)\}$ be in $\mathbf{y}$. Now $\{y(n)\}$ is a Cauchy sequence of rational numbers, so for $\varepsilon = 1$ there is natural number $N(1)$ such that for all $m, n > N(1)$ we have $|y(n) - y(m)| < \varepsilon = 1$. Then $\{y(n)\}$ is bounded above by the rational number $\max\{y(1), y(2), \ldots, y(N(1)), y(N(1)+1)\}$. Let $b = \max\{y(1), y(2), \ldots, y(N(1)), y(N(1)+1)\} + 1$ and let $\mathbf{b}$ be the equivalence class containing $\{b, b, \ldots\}$. Then $\mathbf{y} < \mathbf{b}$, where $\mathbf{b}$ is rational.

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Theorem 2.1.A (continued)

Theorem 2.1.A. The ordered field of real numbers $\mathbb{R}$ is Archimedean.

Proof (continued). So for any positive real numbers $\mathbf{x}$ and $\mathbf{y}$, we have rational $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{0} < \mathbf{a} < \mathbf{x}$ and $\mathbf{y} < \mathbf{b}$.

The rational numbers form an Archimedean field by Lemma 1.4.A, so for positive rational numbers a and b, we have that there is a natural number n such that $na > b$ (or $na - b$ is positive). For sequences $\{n, n, \ldots\}$, $\{a, a, \ldots\}$, and $\{b, b, \ldots\}$ we have
$\{n, n, \ldots\} \cdot \{a, a, \ldots\} - \{b, b, \ldots\} = \{na - b, na - b, \ldots\}$ is positive. Therefore $\mathbf{na} - \mathbf{b}$ is positive, or $\mathbf{na} > \mathbf{b}$. Similarly $\mathbf{na} < \mathbf{nx}$, and so by transitivity of the ordering we have $\mathbf{y} < \mathbf{b} < \mathbf{na} < \mathbf{nx}$. That is, for any positive real numbers $\mathbf{x}$ and $\mathbf{y}$, there is a natural number $\mathbf{n}$ such that $\mathbf{nx} > \mathbf{y}$ so that $\mathbb{R}$ is Archimedean, as claimed. $\square$

Theorem 2.1.A (continued)

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Theorem 2.1.7

Theorem 2.1.7. The real numbers $\mathbb{R}$ form an order complete ordered field.

Proof. We need to show that an arbitrary Cauchy sequence of real numbers converges to a real number. Let $\{\mathbf{x}(n)\}$ be a Cauchy sequence of real numbers. We want to show that this sequence converges to a real number $\mathbf{b}$. We do so by finding a Cauchy sequence of rational numbers $\{b_n\}_{n=1}^{\infty}$ and then consider the equivalence class $\mathbf{b}$ containing $\{b_n\}_{n=1}^{\infty}$.

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For each $n \in \mathbb{N}$ let $\{x(n,i)\}_{i=1}^{\infty}$ be a representative of $\mathbf{x}(n)$. Then $\{x(n,i)\}_{i=1}^{\infty}$ is a Cauchy sequence of rational numbers, so there exists natural number N_n such that for all $j, k > N_n$ we have $|x(n,j) - x(n,k)| < 1/n$. Define $b_n = x(n, N_n + 1)$, so that

$$|x(n,i) - b_n| < 1/n \text{ for all } i > N_n. \qquad (1)$$

Then $\{b_n\}$ is a sequence of rational numbers.

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Theorem 2.1.7 (continued 1)

Theorem 2.1.7. The real numbers $\mathbb{R}$ form an order complete ordered field.

Proof (continued). Next, we show that $\{b_n\}_{n=1}^\infty$ is Cauchy. Let $\varepsilon > 0$ where ε is rational (we consider a rational ε since, by definition, this is what is needed to show a sequence of rationals is Cauchy). Since ordered field $\mathbb{Q}$ is Archimedean by Lemma 1.4.A, there is natural number $N^* = N^*(\varepsilon)$ such that

$$1/N^* < \varepsilon/3 \qquad (2)$$

Since $\{\mathbf{x}(n)\}_{n=1}^\infty$ is a Cauchy sequence of real numbers by hypothesis, then with $\boldsymbol{\varepsilon}/\mathbf{3} = (\varepsilon/3, \varepsilon/3, \ldots)$ we have by Note RU.K that there exists natural number $N^{**} = N^{**}(\varepsilon)$ such that for all $m, n > N^{**}$ there are natural numbers M_c and N_c (dependent on m and n) where for all $i > N_c$ we have $\varepsilon/3 - |x(n,i) - x(m,i)| > 1/M_c$ or

$$|x(n,i) - x(m,i)| < \varepsilon/3 - 1/M_c < \varepsilon/3. \qquad (3)$$

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$$1/N^* < \varepsilon/3 \qquad (2)$$

Since $\{\mathbf{x}(n)\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers by hypothesis, then with $\boldsymbol{\varepsilon}/\mathbf{3} = (\varepsilon/3, \varepsilon/3, \ldots)$ we have by Note RU.K that there exists natural number $N^{**} = N^{**}(\boldsymbol{\varepsilon})$ such that for all $m, n > N^{**}$ there are natural numbers M_c and N_c (dependent on m and n) where for all $i > N_c$ we have $\varepsilon/3 - |x(n,i) - x(m,i)| > 1/M_c$ or

$$|x(n,i) - x(m,i)| < \varepsilon/3 - 1/M_c < \varepsilon/3. \qquad (3)$$

Theorem 2.1.7 (continued 2)

Theorem 2.1.7. The real numbers $\mathbb{R}$ form an order complete ordered field.

Proof (continued). Let $N = \max\{N^*, N^{**}\}$ and suppose $m, n > N$. (Notice that N depends on ε and not on m and/or n.) Since $m, n > N^{**}$ then there exist natural numbers M' and N' such that for all $i > N'$ we have by (3) that

$$|x(n,i) - x(m,i)| < \varepsilon/3 - 1/M' < \varepsilon/3. \qquad (4)$$

For $i > N_n$ we have by (1) that

$$|b_n - x(n,i)| < 1/n \qquad (5)$$

and for $i > N_m$ we have by (1) that

$$|b_m - x(m,i)| < 1/m. \qquad (6)$$

Theorem 2.1.7 (continued 2)

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Theorem 2.1.7 (continued 3)

Proof (continued). So for any $i > \max\{N', N_n, N_m\}$ (notice the value of i depends on m and n) we have

$$\begin{aligned}
|b_n - b_m| &= |b_n - x(n,i) + x(n,i) - x(m,i) + x(m,i) - b_m| \\
&\le |b_n - x(n,i)| + |x(n,i) - x(m,i)| + |x(m,i) - b_m| \\
&\quad \text{ by the Triangle Inequality in } \mathbb{Q} \\
&< 1/n + \varepsilon/3 + 1/m \text{ by (5) (since } i > N_n\text{),} \\
&\quad \text{(4) (since } m, n > N^{**} \text{ and } i > N'\text{), and} \\
&\quad \text{(6) (since } i > N_m\text{), respectively)} \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \text{ by (2) since } m, n > N^*.
\end{aligned}$$

That is, for all $m, n > N$ we have $|b_n - b_m| < \varepsilon$. Therefore, $\{b_n\}_{n=1}^{\infty}$ is a Cauchy sequence of rational numbers. Hence, the equivalence class containing $\{b_n\}_{n=1}^{\infty}$, $\mathbf{b}$, is a real number.

Theorem 2.1.7 (continued 3)

Proof (continued). So for any $i > \max\{N', N_n, N_m\}$ (notice the value of i depends on m and n) we have

$$\begin{array}{rcl} |b_n - b_m| & = & |b_n - x(n,i) + x(n,i) - x(m,i) + x(m,i) - b_m| \\ & \leq & |b_n - x(n,i)| + |x(n,i) - x(m,i)| + |x(m,i) - b_m| \\ & & \text{ by the Triangle Inequality in } \mathbb{Q} \\ & < & 1/n + \varepsilon/3 + 1/m \text{ by (5) (since } i > N_n\text{),} \\ & & \text{(4) (since } m, n > N^{**} \text{ and } i > N'\text{), and} \\ & & \text{(6) (since } i > N_m\text{), respectively)} \\ & < & \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \text{ by (2) since } m, n > N^*. \end{array}$$

That is, for all $m, n > N$ we have $|b_n - b_m| < \varepsilon$. Therefore, $\{b_n\}_{n=1}^{\infty}$ is a Cauchy sequence of rational numbers. Hence, the equivalence class containing $\{b_n\}_{n=1}^{\infty}$, $\mathbf{b}$, is a real number.

Theorem 2.1.7 (continued 4)

Theorem 2.1.7. The real numbers $\mathbb{R}$ form an order complete ordered field.

Proof (continued). To prove $\mathbf{b}$ is the limit of $\{\mathbf{x}(n)\}_{n=1}^{\infty}$, let $\varepsilon > \mathbf{0}$ where ε is a real number (we consider a real ε since this is needed to show convergence of a sequence of real numbers). Let $\{e(i)\}_{i=1}^{\infty}$ be a representative of ε. Since ε is positive, then by the definition of a positive Cauchy sequence of rational numbers. So (by definition) there are natural numbers M' and N' so that for all $i > N'$ we have $e(i) > 1/M'$. Consider the constant sequence $\{1/M'\}_{i=1}^{\infty}$ as a representative of real number $\mathbf{1}/\mathbf{M}'$. We now have $\mathbf{1}/\mathbf{M}' < \varepsilon$.

As above, let $\{x(n,i)\}_{i=1}^{\infty}$ be a representative of $\mathbf{x}(n)$. Let $N(\varepsilon) = 2M'$. Then for all $n > N(\varepsilon)$ we have by (1) that there is a natural number N_n such that if $i > N_n$ then $|x(n,i) - b_n| < 1/n < 1/(2M')$.

Theorem 2.1.7 (continued 4)

Theorem 2.1.7. The real numbers $\mathbb{R}$ form an order complete ordered field.

Proof (continued). To prove $\mathbf{b}$ is the limit of $\{\mathbf{x}(n)\}_{n=1}^{\infty}$, let $\varepsilon > \mathbf{0}$ where ε is a real number (we consider a real ε since this is needed to show convergence of a sequence of real numbers). Let $\{e(i)\}_{i=1}^{\infty}$ be a representative of ε. Since ε is positive, then by the definition of a positive Cauchy sequence of rational numbers. So (by definition) there are natural numbers M' and N' so that for all $i > N'$ we have $e(i) > 1/M'$. Consider the constant sequence $\{1/M'\}_{i=1}^{\infty}$ as a representative of real number $\mathbf{1}/\mathbf{M}'$. We now have $\mathbf{1}/\mathbf{M}' < \varepsilon$.

As above, let $\{x(n,i)\}_{i=1}^{\infty}$ be a representative of $\mathbf{x}(n)$. Let $N(\varepsilon) = 2M'$. Then for all $n > N(\varepsilon)$ we have by (1) that there is a natural number N_n such that if $i > N_n$ then $|x(n,i) - b_n| < 1/n < 1/(2M')$.

Theorem 2.1.7 (continued 5)

Theorem 2.1.7. The real numbers $\mathbb{R}$ form an order complete ordered field.

Proof (continued). That is, for all $n > N(\varepsilon)$ there are natural numbers M and N, namely $N = N_n$ and $M = 2M'$, such that if $i > N$ then $|x(n,i) - b_n| < 1/(2M') = 1/M' - 1/(2M')$ or $1/M' = |x(n,i) - b_n| > 1/(2M')$ and hence $e(i) = |x(n,i) - b_n| > 1/M' = |x(n,i) - b_n| > 1/(2M') = 1/M$. By Note RU.L, this gives (in terms of real numbers) $|\mathbf{x}(n) - \mathbf{b}| < \mathbf{1}/\mathbf{M}' < \varepsilon$. Therefore $\mathbf{x}(n)$ converges to $\mathbf{b}$.

Since $\mathbf{x}(n)$ is an arbitrary Cauchy sequence of real numbers, then every Cauchy sequence of real numbers converges. That is, $\mathbb{R}$ is Cauchy complete. By Theorem 1.2.A, $\mathbb{R}$ is Archimedean, so by Theorem 1.4.3 we have that $\mathbb{R}$ is order complete, as claimed. □

Theorem 2.1.7 (continued 5)

Theorem 2.1.7. The real numbers $\mathbb{R}$ form an order complete ordered field.

Proof (continued). That is, $\underline{\text{for all } n > N(\varepsilon)}$ there are natural numbers M and N, namely $N = N_n$ and $M = 2M'$, such that if $i > N$ then $|x(n,i) - b_n| < 1/(2M') = 1/M' - 1/(2M')$ or $1/M' = |x(n,i) - b_n| > 1/(2M')$ and hence $e(i) = |x(n,i) - b_n| > 1/M' = |x(n,i) - b_n| > 1/(2M') = 1/M$. By Note RU.L, this gives (in terms of real numbers) $|\mathbf{x}(n) - \mathbf{b}| < \mathbf{1}/\mathbf{M}' < \varepsilon$. Therefore $\mathbf{x}(n)$ converges to $\mathbf{b}$.

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Theorem 1.3.1

Theorem 1.3.1. The function $i : \mathbb{N} \cup \{0\} \to S$, where S is an ordered field, satisfies:

(a) $i(n+m) = i(n) + i(m)$ for all $m, n \in \mathbb{N} \cup \{0\}$,

(b) $i(nm) = i(n)i(m)$ for all $m, n \in \mathbb{N} \cup \{0\}$, and

(c) i is one to one on $\mathbb{N} \cup \{0\}$.

Proof. **(a)** This holds trivially for $n = 0$. We give an inductive proof on n. Let $m \in \mathbb{N}$ be arbitrary but fixed. For the base case $n = 1$, we have

$$\begin{aligned} i(1+m) &= \underbrace{1_S + 1_S + \cdots + 1_S + 1_S}_{1+m \text{ times}} = 1_S + (\underbrace{1_S + 1_S + \cdots + 1_S}_{m \text{ times}}) \\ &= i(1) + i(m) \text{ by the definition of } i. \end{aligned}$$

For the induction hypothesis, suppose for $n = k \geq 1$ we have $i(k+m) = i(k) + i(m)$.

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For the induction hypothesis, suppose for $n = k \geq 1$ we have $i(k+m) = i(k) + i(m)$.

Theorem 1.3.1 (continued 1)

Theorem 1.3.1. The function $i : \mathbb{N} \cup \{0\} \to S$, where S is an ordered field, satisfies:

(a) $i(n+m) = i(n) + i(m)$ for all $m, n \in \mathbb{N} \cup \{0\}$.

Proof (continued). Now consider:

$$\begin{aligned} i((k+1)+m) &= i((k+m)+1) = i(k+m_{+}1_S \text{ by the base case,} \\ & \text{where } m \text{ is replaced with } k+m \\ &= (i(k)+i(m)) + 1_S \text{ by the induction hypothesis} \\ &= (i(k)+1_S) + i(m) = i(k+1) + i(m) \text{ by the} \\ & \text{base case where } m \text{ is replaced with } k+1. \end{aligned}$$

So the result holds for $n = k+1$ giving the induction step. Therefore, the claim holds for all $n \in \mathbb{N}$ by mathematical induction and, since $m \in \mathbb{N}$ is arbitrary, $i(n+m) = i(n) + i(m)$ for all $m, n \in \mathbb{N}$, as claimed.

Theorem 1.3.1 (continued 2)

Theorem 1.3.1. The function $i : \mathbb{N} \cup \{0\} \to S$, where S is an ordered field, satisfies:

(b) $i(nm) = i(n)i(m)$ for all $m, n \in \mathbb{N} \cup \{0\}$.

Proof (continued). (b) Again, we give a inductive proof on n. Let $m \in \mathbb{N}$ be arbitrary but fixed. For the base case $n = 1$, we have $i(1m) = i(m) = 1_S i(m) = i(1)i(m)$. For the inductive hypothesis, suppose for $n = k \geq 1$ we have $i(km) = i(k)i(m)$. Now consider:

$$\begin{aligned} i((k+1)m) &= i(km+m) = i(km) + i(m) \text{ by part (a)} \\ &= i(k)i(m) + i(m) \text{ by the induction hypothesis} \\ &= i(k+1)i(m(\text{ by part (a)}. \end{aligned}$$

So the result holds for $n = k + 1$, giving the induction step . Therefore, the claim holds for all $n \in \mathbb{N}$ by mathematical induction and, since $m \in \mathbb{N}$ is arbitrary, $i(nm) = i(n)i(m)$ for all $m, n \in \mathbb{N}$, as claimed.

Theorem 1.3.1 (continued 3)

Theorem 1.3.1. The function $i : \mathbb{N} \cup \{0\} \to S$, where S is an ordered field, satisfies:

(c) i is one to one on $\mathbb{N} \cup \{0\}$.

Proof (continued). (c) To show i is one to one, suppose $i(m) = i(n)$ for some $m, n \in \mathbb{N}$ where, WLOG, say $n \geq m$. Then

$$i(n) = i(n - m + m) = i(n - m) + i(m) \text{ or } i(n) - i(m) = i(n - m)$$

or (since $i(n) = i(m)$) $0_S = i(n - m)$. But the only nonnegative integer mapped mapped to 0_S is 0, so that $n - m = 0$ and $n = m$. That is, i is one to one on $\mathbb{N} \cup \{0\}$, as claimed. □

Theorem 2.3.3

Theorem 2.3.B. (Problem 2.3.1) Mapping $i : \mathbb{Q} \to S$ is well-defined. That is, $i(p/q) = i(r/s)$ for $p/q = r/s$ where $p, q, r, s \in \mathbb{Z}$.

Proof. Notice that rational numbers p/q and r/s (where $p, q, r, s \in \mathbb{Z}$ with $q \neq 0$ and $s \neq 0$) are equal if and only if $ps = qr$. If $r = 0$ then $ps = qr = q(0) = 0$ so that $p = 0$ since $s \neq 0$. Then $p/q = r/s = 0$ and $i(p/q) = i(r/s) = 0_S$. So we assume WLOG that $r \neq 0$. With $p/q = r/s$ we have

$$\begin{aligned} i(p/q) &= r((p/q)(1_S)) = i((p/q)(r/s)(s/r)) \text{ since } r \neq 0 \\ &= i((ps)/(qr) \cdot (r/s)) = i(1_S \cdot (r/s)) \text{ since } ps = qr \\ &= i(r/s). \end{aligned}$$

Hence, the value of i on an element of $\mathbb{Q}$ is independent of the representative of the element. That is, i is well-defined on $\mathbb{Q}$, as claimed. $\square$

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Hence, the value of i on an element of $\mathbb{Q}$ is independent of the representative of the element. That is, i is well-defined on $\mathbb{Q}$, as claimed. $\square$

Theorem 2.3.1

Theorem 2.3.1. The function $i : \mathbb{Q} \to S$ is a field and order isomorphism from the $\mathbb{Q}$ onto a subfield of S.

Proof. The image of i is $S_{\mathbb{Q}} = \{i(p/q) \mid p/q \in \mathbb{Q}\}$. Of course i is onto its image, so i maps $\mathbb{Q}$ onto $S_{\mathbb{Q}}$. Consider $i(p_1/q_1), i(p_2/q_2) \in S_{\mathbb{Q}}$ where $i(p_1/q_1) = i(p_2/q_2)$. Then $i(p_1)(i(q_1))^{-1} = i(p_2)(i(q_2))^{-1}$ or $i(p_1)i(q_2) = i(p_2)i(q_1)$ or $i(p_1 q_2) = i(p_2 q_1)$ by Theorem 1.3.1(b) and Theorem 1.3.2(b). Since i is one to one on $\mathbb{Z}$ by Theorem 1.3.2(c), then $p_1 q_2 = p_2 q_1$, or $p_1/q_1 = p_2/q_2$ so that i is on to one on $\mathbb{Q}$.

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Let $p_1/q_1, p_2/q_2 \in \mathbb{Q}$. Then

$$\begin{aligned} i\left(\frac{p_1}{q_1} \cdot \frac{p_2}{q_2}\right) &= i\left(\frac{p_1 p_2}{q_1 q_2}\right) = i(p_1 p_2)/i(q_1 q_2) \text{ by the definition of } i \\ &= i(p_1)i(p_2)/(i(q_1)i(q_2)) \text{ by Theorems 1.3.1(b) and 1.3.2(b)} \\ &= \frac{i(p_1)}{i(q_1)} \cdot \frac{i(p_2)}{i(q_2)} = i\left(\frac{p_1}{q_1}\right) i\left(\frac{p_2}{q_2}\right) \text{ by the definition of } i. \end{aligned}$$

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Let $p_1/q_1, p_2/q_2 \in \mathbb{Q}$. Then

$$\begin{aligned} i\left(\frac{p_1}{q_1} \cdot \frac{p_2}{q_2}\right) &= i\left(\frac{p_1 p_2}{q_1 q_2}\right) = i(p_1 p_2)/i(q_1 q_2) \text{ by the definition of } i \\ &= i(p_1)i(p_2)/(i(q_1)i(q_2)) \text{ by Theorems 1.3.1(b) and 1.3.2(b)} \\ &= \frac{i(p_1)}{i(q_1)} \cdot \frac{i(p_2)}{i(q_2)} = i\left(\frac{p_1}{q_1}\right) i\left(\frac{p_2}{q_2}\right) \text{ by the definition of } i. \end{aligned}$$

Theorem 2.3.1 (continued 1)

Proof (continued). Also,

$$\begin{aligned}
i\left(\frac{p_1}{q_1}+\frac{p_2}{q_2}\right) &= i\left(\frac{p_1q_2+p_2q_1}{q_1q_2}\right)=\frac{i(p_1q_2+p_2q_1)}{i(q_1q_2)} \text{ by the definition of } i\\
&= (i(p_1q_2)+i(p_2q_1))/(i(q_1)i(q_2)) \text{ by Theorems 1.3.1(a)}\\
&\quad \text{and 1.3.2(a)}\\
&= \frac{i(p_1)i(q_2)+i(p_2)i(q_1)}{i(q_1)i(q_2)} \text{ by Theorems 1.3.1(b)}\\
&\quad \text{and 1.3.2(b)}\\
&= \frac{i(p_1)i(q_2)}{i(q_1)i(q_2)}+\frac{i(p_2)i(q_1)}{i(q_1)i(q_2)}=\frac{i(p_1)}{i(q_1}+\frac{i(p_2)}{i(q_2)}\\
&= i(p_1/q_1)+i(p_2/q_2) \text{ by the definition of } i.
\end{aligned}$$

Therefore $i:\mathbb{Q}\to S_{\mathbb{Q}}$ is a ring isomorphism. Since $\mathbb{Q}$ is a field, then $S_{\mathbb{Q}}$ is a subfield of S and i is actually a field isomorphism.

Theorem 2.3.1 (continued 2)

Proof (continued). Now to show that i is an order isomorphism; that is, i preserves the order. As observed in Note RU.N, i maps positive integers to positive elements of S and maps negative integers to negative elements of S. Notice that for positive n in an ordered field we also have that $n^{-1} = 1/n$ by Kirkwood's Exercise 1.2.7(b) is positive. So for any positive $p/q \in \mathbb{Q}$ (say both p and q are positive integers) we have $i(p/q) = i(p)(i(q))^{-1}$ where $i(p)$ and $i(q)$ are positive. Since S is an ordered field and $i(q)$ is positive, then Kirkwood's Exercise 1.2.7(b) gives that $(i(q))^{-1}$ is positive. So by the closure of the positive set in S, $i(p)(i(q))^{-1} = i(p)/i(q) = i(p/q)$ is positive. Hence i maps all positive rationals to positive elements of S.

Theorem 2.3.1 (continued 2)

Proof (continued). Now to show that i is an order isomorphism; that is, i preserves the order. As observed in Note RU.N, i maps positive integers to positive elements of S and maps negative integers to negative elements of S. Notice that for positive n in an ordered field we also have that $n^{-1} = 1/n$ by Kirkwood's Exercise 1.2.7(b) is positive. So for any positive $p/q \in \mathbb{Q}$ (say both p and q are positive integers) we have $i(p/q) = i(p)(i(q))^{-1}$ where $i(p)$ and $i(q)$ are positive. Since S is an ordered field and $i(q)$ is positive, then Kirkwood's Exercise 1.2.7(b) gives that $(i(q))^{-1}$ is positive. So by the closure of the positive set in S, $i(p)(i(q))^{-1} = i(p)/i(q) = i(p/q)$ is positive. Hence i maps all positive rationals to positive elements of S. Similarly, if $p/q \in \mathbb{Q}$ is negative (say $p < 0$ and $q > 0$) we have $i(p)$ is negative in S (as observed above), so $i(p/q) = i(p)(i(q))^{-1}$ is negative in S by Kirkwood's Theorem 1-7(d) (which implies that a positive times a negative is negative). So the positive set in S corresponds exactly to the positive set in $\mathbb{Q}$ under i. That is, i is a field and order isomorphism, as claimed. □

Theorem 2.3.1 (continued 2)

Proof (continued). Now to show that i is an order isomorphism; that is, i preserves the order. As observed in Note RU.N, i maps positive integers to positive elements of S and maps negative integers to negative elements of S. Notice that for positive n in an ordered field we also have that $n^{-1} = 1/n$ by Kirkwood's Exercise 1.2.7(b) is positive. So for any positive $p/q \in \mathbb{Q}$ (say both p and q are positive integers) we have $i(p/q) = i(p)(i(q))^{-1}$ where $i(p)$ and $i(q)$ are positive. Since S is an ordered field and $i(q)$ is positive, then Kirkwood's Exercise 1.2.7(b) gives that $(i(q))^{-1}$ is positive. So by the closure of the positive set in S, $i(p)(i(q))^{-1} = i(p)/i(q) = i(p/q)$ is positive. Hence i maps all positive rationals to positive elements of S. Similarly, if $p/q \in \mathbb{Q}$ is negative (say $p < 0$ and $q > 0$) we have $i(p)$ is negative in S (as observed above), so $i(p/q) = i(p)(i(q))^{-1}$ is negative in S by Kirkwood's Theorem 1-7(d) (which implies that a positive times a negative is negative). So the positive set in S corresponds exactly to the positive set in $\mathbb{Q}$ under i. That is, i is a field and order isomorphism, as claimed. □

Theorem 2.3.3

Theorem 2.3.3. Every order complete ordered field is isomorphic to $\mathbb{R}$ (where we take $\mathbb{R}$ to be the complete ordered field of equivalence classes of Cauchy sequences of rational numbers).

Proof. Let S be a complete, ordered field. We will define an isomorphism $i : \mathbb{R} \to S$. On $\mathbb{Q} \subset \mathbb{R}$, we define i as above. We take $\mathbb{R}$ as the "Cantor reals" of Cauchy sequences of real numbers. So the equivalence class of rational numbers which converge to $p/q \in \mathbb{Q}$, $\mathbf{p}/\mathbf{q}$, is mapped by i into S as $\mathbf{p}/\mathbf{q} \mapsto i(p)(i(q))^{-1} = i(p)/i(q)$.

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For $\mathbf{x}$ any real number, we take $\{x(n)\}_{n=1}^{\infty}$ a representative of $\mathbf{x}$. Then $\{x(n)\}_{n=1}^{\infty}$ is a Cauchy sequence of rational numbers. Consider the sequence $\{i(x(n))\}_{n=1}^{\infty}$ in S. Let $\varepsilon \in S$ be positive (we consider a positive element of S since we want to show convergence in S). An order complete field is Archimedean by Theorem 1.4.3, so there is a natural number $k \in S_{\mathbb{N}}$ such that $1/k < \varepsilon$. With $k \in \mathbb{N}$, $1/k \in \mathbb{R}$ satisfies $i(1/k) = 1/k$.

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Theorem 2.3.3 (continued 1)

Proof (continued). Since $\{x(n)\}_{n=1}^{\infty}$ is a Cauchy sequence of rational numbers, then there is $N(1/k) \in \mathbb{N}$ such that for all $m, n > N$ we have $|x(n) = x(m)| < 1/k$. Since $i : \mathbb{Q} \to S$ is an order isomorphism by Theorem 2.3.1, then for all $m, n > N$ we have $|i(x(m)) - i(x(n))| < i(1/k) = 1/k < \varepsilon$. Therefore $\{i(x(n))\}_{n=1}^{\infty}$ is a Cauchy sequence in S. Since S is order complete then it is Cauchy complete by Theorem 1.4.3, so $\{i(x(n))\}_{n=1}^{\infty}$ converges to an element of S. Define $i(\mathbf{x}) = \lim_{n\to\infty} i(x(n))$. Then $i : \mathbb{R} \to S$.

We now show that i is well-defined. Let $\{x(n)\}_{n=1}^{\infty}, \{x'(n)\}_{n=1}^{\infty} \in \mathbf{x}$. Then by the definition of "equivalent sequences of rational numbers," $\{x(n) - y(x)\}_{n=1}^{\infty}$ is a null sequence so that $\lim_{n\to\infty}(x(n_= x'(n)) = 0$. So for all natural numbers k there is $N(k) = N \in \mathbb{N}$ such that for all $n > N$ we have $-1/k < x(n) - x'(n) < 1/k$ (since $\mathbb{R}$ is Archimedean by Theorem 2.1.A).

Theorem 2.3.3 (continued 1)

Proof (continued). Since $\{x(n)\}_{n=1}^{\infty}$ is a Cauchy sequence of rational numbers, then there is $N(1/k) \in \mathbb{N}$ such that for all $m, n > N$ we have $|x(n) = x(m)| < 1/k$. Since $i : \mathbb{Q} \to S$ is an order isomorphism by Theorem 2.3.1, then for all $m, n > N$ we have $|i(x(m)) - i(x(n))| < i(1/k) = 1/k < \varepsilon$. Therefore $\{i(x(n))\}_{n=1}^{\infty}$ is a Cauchy sequence in S. Since S is order complete then it is Cauchy complete by Theorem 1.4.3, so $\{i(x(n))\}_{n=1}^{\infty}$ converges to an element of S. Define $i(\mathbf{x}) = \lim_{n\to\infty} i(x(n))$. Then $i : \mathbb{R} \to S$.

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Theorem 2.3.3 (continued 2)

Proof (continued). Since $i : \mathbb{Q} \to S$ is an order isomorphism by Theorem 2.3.1, for all $n > N$ we have $i(-1/k) < i(x(n)) - i(x'(n)) < i(1/k)$ or $|i * x(n)) - i(x'(n))| < i(1/k)$. Since S is Archimedean by Theorem 1.4.3, then $\lim_{n\to\infty}(x(n) - x'(n)) = 0$ and, since $\lim_{n\to\infty} x(n)$ and $\lim_{n\to infty} x'(n)$ both exist as shown above, $i(\mathbf{x}_= \lim_{n\to\infty} x(n) = \lim_{n\to\infty} x'(n)$. Therefore $i : \mathbb{R} \to S$ is well-defined.

Next, we show that i is one to one. Suppose $i(\mathbf{x}) = i(\mathbf{x}')$. Let $x(n)\}_{n=1}^{\infty}$ be a representative of $\mathbf{x}$ and let $\{x'(n)\}_{n=1}^{\infty}$ be a representative of $\mathbf{x}'$. Then $i(\mathbf{x}) = \lim_{n\to\infty} i(x(n)) = \lim_{n\to\infty} i(x'(n)) = i(\mathbf{x}')$. So $\lim_{n\to\infty} i(x(n)) - \lim_{n\to\infty} i(x'(n)) = \lim_{n\to\infty}(i(x(n)) - i(x'(n)) = 0$. So for all natural numbers $k \in S_{\mathbb{N}}$, there is natural number $N(k) = N \in \mathbb{N}$ such that for all $n < N$ we have $|i(x(n)) - i(x'(n))| < 1/k$. Since $i : \mathbb{Q} \to S$ is an order isomorphism by Theorem 2.3.1, for all $n > N$ we have $|x(n) - x'(n)| < 1/k$.

Theorem 2.3.3 (continued 2)

Proof (continued). Since $i : \mathbb{Q} \to S$ is an order isomorphism by Theorem 2.3.1, for all $n > N$ we have $i(-1/k) < i(x(n)) - i(x'(n)) < i(1/k)$ or $|i * x(n)) - i(x'(n))| < i(1/k)$. Since S is Archimedean by Theorem 1.4.3, then $\lim_{n\to\infty}(x(n) - x'(n)) = 0$ and, since $\lim_{n\to\infty} x(n)$ and $\lim_{n\to infty} x'(n)$ both exist as shown above, $i(\mathbf{x}_= \lim_{n\to\infty} x(n) = \lim_{n\to\infty} x'(n)$. Therefore $i : \mathbb{R} \to S$ is well-defined.

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Theorem 2.3.3 (continued 3)

Proof (continued). Since $\mathbb{R}$ is Archimedean by Theorem 2.1.A then $\lim_{n\to\infty}(x(n) - x'(n)) = 0$ or $\{x(n) - x'(n)\}_{n=1}^{\infty}$ is a null sequence and so, by the definition of equivalent sequences of rational numbers, $x \sim x'$ or $\mathbf{x} = \mathbf{x}'$. Hence, i is one to one.

We now show that i is onto. Let $s \in S$. Since $S_{\mathbb{Q}}$ is dense in S by Theorem 2.3.2, there is a sequence of rational in S that converge to S (for each $k \in \mathbb{N}$ there is $q(k) \in S_{\mathbb{Q}}$ in the interval $(s - 1/k, s + 1/k)$ and the sequence $\{q(k)\} \to s$). Since i maps $\mathbb{Q}$ onto $S_{\mathbb{Q}}$ (this is the definition of $S_{\mathbb{Q}}$), then there are rationals $p(k) \in \mathbb{Q}$ such that $i(p(k)) = q(k)$. Then $\{p(k)\}_{k=1}^{\infty}$ is a Cauchy sequence of rational numbers (similar to the argument above in the proof that i is one to one). The real number $\mathbf{x}$ which the equivalence class containing $\{p(k\}_{k=1}^{\infty}$ is mapped by i to s, $i(\mathbf{x}) = s$. Therefore, i is onto.

Theorem 2.3.3 (continued 3)

Proof (continued). Since $\mathbb{R}$ is Archimedean by Theorem 2.1.A then $\lim_{n\to\infty}(x(n) - x'(n)) = 0$ or $\{x(n) - x'(n)\}_{n=1}^{\infty}$ is a null sequence and so, by the definition of equivalent sequences of rational numbers, $x \sim x'$ or $\mathbf{x} = \mathbf{x}'$. Hence, i is one to one.

We now show that i is onto. Let $s \in S$. Since $S_{\mathbb{Q}}$ is dense in S by Theorem 2.3.2, there is a sequence of rational in S that converge to S (for each $k \in \mathbb{N}$ there is $q(k) \in S_{\mathbb{Q}}$ in the interval $(s - 1/k, s + 1/k)$ and the sequence $\{q(k)\} \to s$). Since i maps $\mathbb{Q}$ onto $S_{\mathbb{Q}}$ (this is the definition of $S_{\mathbb{Q}}$), then there are rationals $p(k) \in \mathbb{Q}$ such that $i(p(k)) = q(k)$. Then $\{p(k)\}_{k=1}^{\infty}$ is a Cauchy sequence of rational numbers (similar to the argument above in the proof that i is one to one). The real number $\mathbf{x}$ which the equivalence class containing $\{p(k\}_{k=1}^{\infty}$ is mapped by i to s, $i(\mathbf{x}) = s$. Therefore, i is onto.

Theorem 2.3.3 (continued 4)

Theorem 2.3.3. Every order complete ordered field is isomorphic to $\mathbb{R}$ (where we take $\mathbb{R}$ to be the complete ordered field of equivalence classes of Cauchy sequences of rational numbers).

Proof (continued). Now, to establish that i is a field isomorphism. Let $\mathbf{x}$ and $\mathbf{y}$ be real numbers, and let $\{x(n)\}_{n=1}^{\infty}$ be a representative of $\mathbf{x}$ and let $\{y(n)\}_{n=1}^{\infty}$ be a representative of $\mathbf{y}$. Then $\{x(n)+y(n)\}_{n=1}^{\infty}$ is a representative of $\mathbf{x}+\mathbf{y}$. So

$$\begin{aligned}
i(\mathbf{x}+\mathbf{y}) &= \lim_{n\to\infty} i(x(n)+y(n)) \\
&= \lim_{n\to\infty} (i(x(n))+i(y(n))) \text{ since } i:\mathbb{Q}\to S \text{ is a field} \\
&\quad \text{isomorphism by Theorem 2.3.B} \\
&= \lim_{n\to\infty} i(x(n)) + \lim_{n\to\infty} i(y(n)) \text{ since both limits exist,} \\
&\quad \text{as shown above when } i \text{ is defined} \\
&= i(\mathbf{x})+i(\mathbf{y}) \text{ by the definition of } i.
\end{aligned}$$

Theorem 2.3.3 (continued 5)

Proof (continued). Also, $\{x(n)y(n)\}_{n=1}^{\infty}$ is a representation of **xy**. So

$$\begin{aligned} i(\mathbf{xy}) &= \lim_{n\to\infty} i(x(n)y(n)) \\ &= \lim_{n\to\infty} i(x(n)) \lim_{n\to\infty} i(y(n)) \text{ since } i:\mathbb{Q}\to S \text{ is a field} \\ &\quad\ \text{isomorphism by Theorem 2.3.B} \\ &= \lim_{n\to\infty} i(x(n)) \lim_{n\to\infty} i(y(n)) \text{ since both limits exist} \\ &= i(\mathbf{x})i(\mathbf{y}) \text{ by the definition of } i. \end{aligned}$$

Therefore i is a field isomorphism.

Finally, we show that i is an order isomorphism. Let $\mathbf{x} \in \mathbb{R}$ be positive and let $\{x(n)\}_{n=1}^{\infty}$ be a representative of **x**. Then (by definition of positive real number) there are natural numbers M and N where for $n > N$ we have $x(n) > 1/M$. Since $i:\mathbb{Q}\to S$ is an order isomorphism then $i(x(n)) > 1_S/i(M) > 0_S$ for $n > N$. So $i(\mathbf{x}) = \lim_{n\to\infty} i(x(n)) \geq 1_S/i(M) > 0_S$, and $i(\mathbf{x})$ is positive.

Theorem 2.3.3 (continued 5)

Proof (continued). Also, $\{x(n)y(n)\}_{n=1}^{\infty}$ is a representation of $\mathbf{xy}$. So

$$\begin{aligned} i(\mathbf{xy}) &= \lim_{n\to\infty} i(x(n)y(n)) \\ &= \lim_{n\to\infty} i(x(n)) \lim_{n\to\infty} i(y(n)) \text{ since } i : \mathbb{Q} \to S \text{ is a field} \\ &\ \text{isomorphism by Theorem 2.3.B} \\ &= \lim_{n\to\infty} i(x(n)) \lim_{n\to\infty} i(y(n)) \text{ since both limits exist} \\ &= i(\mathbf{x})i(\mathbf{y}) \text{ by the definition of } i. \end{aligned}$$

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Theorem 2.3.3 (continued 6)

Theorem 2.3.3. Every order complete ordered field is isomorphic to $\mathbb{R}$ (where we take $\mathbb{R}$ to be the complete ordered field of equivalence classes of Cauchy sequences of rational numbers).

Proof (continued). If $\mathbf{x} < \mathbf{0}$ then $-\mathbf{x} > \mathbf{0}$ and we now have $i(-\mathbf{x}) > i(\mathbf{0}) = 0_S$ Since $i(-\mathbf{x}) = -i(\mathbf{x})$ (because i is a field isomorphism), so $-i(\mathbf{x}) > 0$ and $i(\mathbf{x}) < 0$. Therefore $\mathbf{x} \in \mathbb{R}$ is positive if and only if $i(\mathbf{x}) \in S$ is positive. So i is an order isomorphism. That is, $i : \mathbb{R} \to S$ is a field and order isomorphism.

Since S is an arbitrary order complete ordered field, then every order complete ordered field is isomorphic to $\mathbb{R}$, as claimed. □

Theorem 2.3.3 (continued 6)

Theorem 2.3.3. Every order complete ordered field is isomorphic to $\mathbb{R}$ (where we take $\mathbb{R}$ to be the complete ordered field of equivalence classes of Cauchy sequences of rational numbers).

Proof (continued). If $\mathbf{x} < \mathbf{0}$ then $-\mathbf{x} > \mathbf{0}$ and we now have $i(-\mathbf{x}) > i(\mathbf{0}) = 0_S$ Since $i(-\mathbf{x}) = -i(\mathbf{x})$ (because i is a field isomorphism), so $-i(\mathbf{x}) > 0$ and $i(\mathbf{x}) < 0$. Therefore $\mathbf{x} \in \mathbb{R}$ is positive if and only if $i(\mathbf{x}) \in S$ is positive. So i is an order isomorphism. That is, $i : \mathbb{R} \to S$ is a field and order isomorphism.

Since S is an arbitrary order complete ordered field, then every order complete ordered field is isomorphic to $\mathbb{R}$, as claimed. □