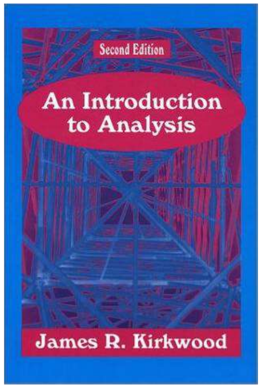
Analysis 1

Chapter 3. Topology of the Real Numbers

Supplement: A Classification of Open Sets of Real Numbers—Proofs of Theorems



Exercise 3.1.4

Exercise 3.1.4. Let A be an open set containing x. Let $\alpha = \sup\{\delta \mid [x, x+\delta) \subset A\}$ and $\beta = \sup\{\delta \mid (x-\delta, x] \subset A\}$. Assume α and β are finite. Then $(x-\beta, x+\alpha) \subset A$, $x+\alpha \notin A$ and $x-\beta \notin A$.

Proof. Since α is the supremum of the set $\{\delta \mid [x, x+\delta) \subset A\}$, there is a sequence $\{\delta_i\}$ of elements of the set such that $\lim \delta_i = \alpha$ by Theorem 2-8.

Since $\delta_i \leq \alpha$ for each $i \in \mathbb{N}$, the $[x, x+\delta_i) \subset [x, x+\alpha)$ for each $i \in \mathbb{N}$. Hence $\cup_{i=1}^{\infty}[x, x+\delta_i) \subset [x, x+\alpha)$. If $y \in [x, x+\alpha)$, then $y < \alpha$ and $\alpha - y > 0$. Since $\{\delta_i\} \to \alpha$, by the definition of limit of a sequence there is some $N \in \mathbb{R}$ such that for all $i > N$ we have $|\alpha - \delta_i| = \alpha - \delta_i < \alpha - y$. That is, $y < \delta_i$ for all $i > N$. Therefore $y \in [x, x+\delta_i)$ for some $i \in \mathbb{N}$. Since y is an arbitrary element of $[x, x+\alpha)$, we have $[x, x+\alpha) \subset \cup_{i=1}^{\infty}[x, x+\delta_i)$. We now have

$$[x, x+\alpha) = \cup_{i=1}^{\infty}[x, x+\delta_i) \subset A.$$

Exercise 3.1.4 (continued)

Exercise 3.1.4. Let A be an open set containing x. Let $\alpha = \sup\{\delta \mid [x, x+\delta) \subset A\}$ and $\beta = \sup\{\delta \mid (x-\delta, x] \subset A\}$. Assume α and β are finite. Then $(x-\beta, x+\alpha) \subset A$, $x+\alpha \notin A$ and $x-\beta \notin A$.

Proof (continued). Similarly, there exists a sequence $\{\delta_j\}$ such that $\lim \delta_j = \beta$ and $(x-\beta, x] = \cup_{j=1}^{\infty}(x-\delta_j, x] \subset A$. So $(x-\beta, x+\alpha) \subset A$.

Now suppose $x+\alpha \in A$. Since A is open then, by the definition of open set, there exists $\Delta > 0$ such that $(x+\alpha-\Delta, x+\alpha+\Delta) \subset A$. But then $[x, x+(\alpha+\Delta)) \subset A$ and $\alpha+\Delta \in \{\delta \mid [x, x+\delta) \subset A\}$, contradicting the fact that α is the l.u.b. of this set. Therefore $x+\alpha \notin A$, and similarly, $x-\beta \notin A$. □

Exercise 3.1.5

Exercise 3.1.5. Let A be an open set. Let I_x be the maximal open interval containing x. Then either $I_x = I_y$ or $I_x \cap I_y = \varnothing$.

Proof. Either $I_x \cap I_y = \varnothing$, or there is some $z \in I_x \cap I_y$.

If $I_x = (x_1, x_2)$, $I_y = (y_1, y_2)$, and $z \in I_x \cap I_y$ then $x_1 < z < x_2$ and $y_1 < z < y_2$ (we allow the possibilities that x_1 or y_1 is $-\infty$, and that x_2 or y_2 is ∞). So $x_1 < y_2$ and $y_1 < x_2$, and $I_x \cup I_y$ is an open interval containing x (either (x_1, x_2), (y_1, y_2), (y_1, x_2) or (x_1, y_2)). Since I_x and I_y are maximal intervals containing x, we have by Note OS.A that $I_x \cup I_y \subset I_x$ and $I_x \cup I_y \subset I_y$. Therefore, $I_x = I_x \cup I_y = I_y$.

So either $I_x \cap I_y = \varnothing$ or $I_x = I_y$. □

Theorem 3-5

Theorem 3-5. Classification of open sets of real numbers.
A set of real numbers is open if and only if it is a countable union of disjoint intervals.

Proof. First, a countable union of open intervals is open by Theorem 3-2 (in fact, in general, an arbitrary union of open sets is open).

Now for the converse. Let A be an open set. If $x \in A$, then there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq A$. Let

$$\alpha = \sup\{\delta \mid [x, x + \delta) \subseteq A\} \text{ and } \beta = \sup\{\delta \mid (x - \delta, x] \subseteq A\}.$$

If α and β are finite, then $(x - \beta, x + \alpha) \subseteq A$ and $x + \alpha \notin A$ and $x - \beta \notin A$ by Exercise 3.1.4. So for each $x \in A$, we can define this "maximal open interval" $I_x = (x - \beta, x + \alpha)$ which contains x and is a subset of A. Notice that if α or β are not finite, then the maximal interval containing x will be unbounded.

Theorem 3-5 (continued)

Theorem 3-5. A set of real numbers is open if and only if it is a countable union of disjoint intervals.

Proof (continued). If $x, y \in A$ then either $I_x = I_y$ or $I_x \cap I_y = \varnothing$ by Exercise 3.1.5. Each I_x contains a rational number (by Exercise 1.3.4(a), say) and each element of A is in some I_x, so

$$A = \cup_{x \in A} I_x = \cup_{x \in A \cap \mathbb{Q}} I_x.$$

Since the rationals are countable by Note 1.3.D (so that any subset of $\mathbb{Q}$ is countable), then A is a countable union of open intervals, as claimed. □