Analysis 1+, MATH 4217/5217, Fall 2025

Homework 5, 2-2 Subsequences,

2-3 The Bolzano-Weierstrass Theorem

Due Saturday, September 27, at 11:59 p.m.

Write in complete sentences and paragraphs!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the hypotheses, class notes, or textbook. Use the notation and techniques described in the in-class hints (this is part of the instructions!). Do not copy the work of others (including websites or AI generated solutions). If you have any questions, then contact me (`gardnerr@etsu.edu`).

2.2.12 (b) If $\{a_n\}$ converges to L, then there is either an increasing subsequence of $\{a_n\}$ that converges to L or a decreasing subsequence of $\{a_n\}$ that converges to L.

2.2.14. Let $\{a_n\}$ be a sequence such that subsequence $\{a_{2n}\}$ converges to L, and subsequence $\{a_{2n+1}\}$ converges to L. Prove that $\{a_n\}$ converges to L. HINT: This is straightforward, but is tricky in terms of notation. Denote the sequence $\{a_n\}_{n=1}^{\infty}$ as $\{a_m\}_{m=1}^{\infty}$, so that $\{a_{2n}\}_{n=1}^{\infty}$ and $\{a_{2n+1}\}_{n=0}^{\infty}$ are $\{a_m\}_{m=2,4,6,\dots}$ and $\{a_m\}_{m=1,3,5,\dots}$, respectively. Apply the definition of convergence to $\{a_{2n}\}$ and $\{a_{2n+1}\}$ to prove that $\{a_m\} = \{a_n\}$ converges. You will need to distinguish between m even and m odd.

2.3.1. Write a formal proof of Theorem 2-13: Let $\{a_n\}$ be a sequence. Then L is a (finite) subsequential limit of $\{a_n\}$ if and only if L satisfies either of the following:

(*i*) There are infinitely many terms of $\{a_n\}$ equal to L, or

(*ii*) L is a limit point of a set consisting of the terms of $\{a_n\}$.

2.3.4. (Graduate Problem) Prove part (b) of Theorem 2-15: A sequence that is unbounded below has a subsequence that diverges to $-\infty$. HINT: Inductively create such a subsequence, as is done in the proof of Theorem 2-15 part (a).

2.3.12. Prove that if every open interval containing L contains a point of the set A distinct from L, then, for any $\varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains infinitely many points of A.