1.3. The Completeness Axiom.

Note. In this section we give the final axiom in the definition of the real numbers, $\mathbb{R}$. The eight axioms of Section 1.2. Properties of the Real Numbers as an Ordered Field give an ordered field. We have seen several examples of ordered fields: $\mathbb{Q}$, $\mathbb{Q}[\sqrt{2}]$, $\mathbb{A}$, and $\mathbb{R}$. So we need an additional constraint to axiomatically isolate the real numbers. We need a few definitions before stating the last axiom.

Definition. Let A be a subset of an ordered field $\mathbb{F}$. If there exists $b \in \mathbb{F}$ such that $a \leq b$ for all $a \in A$, then b is an *upper bound* of A and A is said to be *bounded above*. If there is a $c \in \mathbb{F}$ such that $c \leq a$ for all $a \in A$, then c is a *lower bound* of A and A is *bounded below*. A set bounded above and below is *bounded*. A set that is not bounded is *unbounded*.

Definition. Let A be a subset of an ordered field $\mathbb{F}$ which is bounded above. Then $b \in \mathbb{F}$ is called a *least upper bound* (*lub* or *supremum*) of a set A if (1) b is an upper bound of A and (2) if c is an upper bound of A, then $b \leq c$.

Definition. Let A be a subset of an ordered field $\mathbb{F}$ which is bounded below. Then $b \in \mathbb{F}$ is called a *greatest lower bound* (*glb* or *infimum*) of a set A if (1) b is an lower bound of A and (2) if c is an lower bound of A, then $c \leq b$.

Note 1.3.A. If set A, a subset of an ordered field, is finite then the lub is simply the greatest element of the set and the glb is simply the least element of the set. In general, the lub and glb of a set may or may not be in the set. For a *nonempty* set of an ordered field, we have $\text{lub}(A) \leq \text{glb}(A)$. In the setting of the real numbers, the least upper bound of the empty set is $\text{lub}(\varnothing) = -\infty$ and the greatest lower bound is $\text{glb}(\varnothing) = +\infty$ (these claims hold vacuously).

Definition. Let $\mathbb{F}$ be an ordered field. $\mathbb{F}$ is *complete* if for any nonempty set $A \subset \mathbb{F}$ that is bounded above, there is a lub of A in $\mathbb{F}$.

Note. We complete our definition of the real numbers with one last axiom:

Axiom 9. Axiom of Completeness. The real numbers are complete.

Note. In Exercise 1.3.7 it is to be shown that the definition of a complete ordered field in terms least upper bounds is equivalent to the statement:

> If S is an ordered field and A is a nonempty subset of S that is bounded below, then A has a greatest lower bound that is an element of S.

Note 1.3.B. The nine axioms of the real numbers consist of seven Field Axioms, the Order Axiom, and the Completeness Axiom. We can concisely say that the real numbers are a *complete ordered field*. In fact, we can say that the real numbers are *the* complete ordered fields, since it can be shown that all complete ordered fields are isomorphic. This is proved in Supplement. The Real Numbers are the Unique Complete Ordered Field, though it requires a knowledge of Cauchy sequences and so this supplement cannot be covered until we complete Chapter 2, "Sequences of Real Numbers."

Note. The rational numbers $\mathbb{Q}$ are not a complete ordered field: $A = \{x \in \mathbb{Q} \mid x^2 < 2\}$ is bounded above (by, say, 2) but there is no least upper bound of A in $\mathbb{Q}$. The algebraic numbers $\mathbb{A}$ are not a complete ordered field: $A = \{x \in \mathbb{A} \mid x < \pi\}$ is bounded above (by, say, 4) but there is no least upper bound of A in $\mathbb{A}$ (since π is "transcendental," not algebraic). The transcendental property of π is due to Johann Lambert (August 26 or 28, 1728–September 25, 1777) and appeared in "Mémoire sur quelques propriétés remarquables des quantités transcendentes circulaires et logarithmiques" ["Memoir on some remarkable properties of circular and logarithmic transcendental quantities"], *Histoire de l'Académie Royale des Sciences et des Belles-Lettres de Berlin* (French), **17**, 265–322 (1768).

Theorem 1-14. If the lub and glb of a set of real numbers exist, then they are unique.

Note. The uniqueness of the lub and glb given by Theorem I.14 allows us to define new functions using these. We now formally define exponentiation of a positive real number with an irrational exponent. Recall that this is done in Calculus 1 using the exponential function where for $x, r \in \mathbb{R}$ with $x > 0$, x^r is defined as $x^r = e^{r \ln x}$ (see my online notes for Calculus 1 on Section 3.8. Derivatives of Inverse Functions and Logarithms). In this class, we do not develop the exponential function until after we have introduced integration (which is used to define the natural logarithm function) in 6-2 Section 6.2. Some Properties and Applications of the Riemann Integral.

Definition. Let $x > 1$ be a positive real number and r a positive irrational number. The x^r is the lub of the set $\{x^p \mid p \in \mathbb{Q}, 0 < p < r\}$. If $x \in (0,1)$, then x^r is $\dfrac{1}{(1/x)^r}$. For $-r$ a negative rational number and $x > 0$, define $x^{-r} = 1/x^r$. For $r = 0$, we define $x^r = x^0 = 1$ for all $x \in \mathbb{R}$ where $x \neq 0$.

Note. The following result introduces ε arguments. You have seen such arguments and definitions throughout the calculus sequence. We will use them in a similar way in this sequence, but first we give a classification of least upper bounds and greatest lower bounds in terms of ε's.

Theorem 1-15.

(a) α is a lub of $A \subset \mathbb{R}$ if and only if

- **(i)** α is an upper bound of A, and
- **(ii)** For all $\varepsilon > 0$ there exists a number $x(\varepsilon) \in A$ such that $x(\varepsilon) > \alpha - \varepsilon$.

(b) β is a glb of $A \subset \mathbb{R}$ if and only if

- **(i)** β is a lower bound of A, and
- **(ii)** For all $\varepsilon > 0$ there exists a number $x(\varepsilon) \in A$ such that $x(\varepsilon) < \beta + \varepsilon$.

Note. As commented in Note 1.3.A, the lub and glb of a set may or may not be in the set. The next theorem gives a property of sets that do not contain their lub. A similar result also holds for sets not containing their glb.

Theorem 1-16. Let $\alpha = \text{lub}(A)$ and suppose $\alpha \notin A$. Then for all $\varepsilon > 0$, the interval $(\alpha - \varepsilon, \alpha)$ contains an infinite number of points of A.

Definition. Let A be a set of real numbers, and suppose c is a real number. Then cA is the set of real numbers $cA = \{cx \mid x \in A\}$.

Note. The next result gives relationships between the lub and glb of sets A and cA. When $c < 0$, for any $x, y \in \mathbb{R}$ with $x < y$ we have $cx > cy$ by Theorem 1-7(d). So when we compare $\text{lub}(A)$, $\text{glb}(A)$, $\text{lub}(cA)$, and $\text{glb}(cA)$, we expect inequalities to be reversed when $c < 0$ so that "least" and "greatest" interchange, and "upper" and "lower" interchange. This is spelled out specifically in part (b) of the next result.

Theorem 1-17. Let A be a bounded set of real numbers, and suppose c is a real number. Then

(a) If $c > 0$: **(i)** $\text{lub}(cA) = c\,\text{lub}(A)$. **(ii)** $\text{glb}(cA) = c\,\text{glb}(A)$.

(b) If $c < 0$: (i) $\text{lub}(cA) = c\,\text{glb}(A)$. **(ii)** $\text{glb}(cA) = c\,\text{lub}(A)$.

Note 1.3.C. The extremely famous Greek mathematician Archimedes of Syracuse (287 BCE–212 BCE) lived and worked in Syracuse, Sicily (but was a mathematician of the "Greek world" of the third century BCE). He likely studied in Alexandria, Egypt with the successors of Euclid. In his *On the Sphere and Cylinder*, he proves that the volume V of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. He states five "assumptions" (or "axioms"), mostly concerning geometric properties. The fifth of his assumptions is:

> "5. Further, of unequal lines, unequal surfaces, and unequal solids, the greater exceeds the less by such a magnitude as, when added to itself, can be made to exceed any assigned magnitude among those which are comparable with [it and with] one another."

This statement of the assumption is based on Thomas Heath's *The Works of Archimedes*, Cambridge University Press (1897) (see his page 4). For more on Archimedes, see the MacTutor biography webpage of Archimedes and my History of Mathematics (MATH 3040) online notes on Section 6.2. Archimedes. The statement in Archimedes' fifth assumption that "when added to itself, can be made to exceed any assigned magnitude" is a claim that any positive real number a can be added to itself a sufficient number of times so that exceeds any given real number b. This modern statement of the claim is known as *The Archimedean Principle* and, formally, is the following. (Notice that the proof depends on the Axiom of Completeness.)

Theorem 1-18. The Archimedean Principle.
If $a, b \in \mathbb{R}$ and $a > 0$, then there is a natural number $n \in \mathbb{N}$ such that $na > b$.

Corollary 1-18. Let a be a positive real number and b any real number. Then there is a natural number n such that $\frac{b}{n} < a$.

Note. With $a = \varepsilon$ and $b = 1$, we can find $n = N$ such that $\dfrac{1}{N} < \varepsilon$. We will use this when dealing with sequences.

Example 1.11. Consider the set

$$A = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \right\} = \left\{ \frac{n}{n+1} \,\middle|\, n \in \mathbb{N} \right\}.$$

Prove the least upper bound of A is 1.

Exercise 1.3.4. **(a)** Between any two real numbers, there is a rational number. **(b)** Between any two real numbers, there is an irrational number.

Note. The remainder of this section is devoted to the cardinalities of sets. You may have covered these topics in Mathematical Reasoning (MATH 3000). See my online notes for Mathematical Reasoning on Section 4.1. Cardinality; Fundamental Counting Principles, Section 4.2. Comparing Sets, Finite or Infinite and Section 4.3. Countable and Uncountable Sets. We start our exploration of cardinalities by defining what it means for two sets to have the "same cardinality." Of course, we wan a definition that works both for finite and infinite sets.

Definition. Two sets A and B are said to have the *same cardinality* if there is a one-to-one and onto function from A to B.

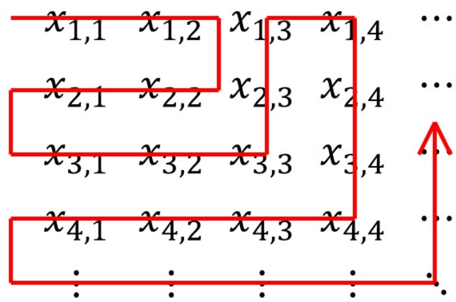
Note. This definition of "same cardinality" is due to Georg Cantor (March 3, 1845–January 6, 1918) in the 1880s. Some biographical information is given on Cantor below. It was recently discovered that Archimedes (287 BCE–212 BCE) was surprisingly close to this ideas as well; see my online notes for History of Mathematics (MATH 3040) on Supplement. Archimedes' *Method*, Part 2 and notice Note AM2.I.

Definition. A set S is said to be *finite with cardinality* n if there is a one-to-one and onto function form S to $\{1, 2, \ldots, n\}$ where $n \in \mathbb{N}$. The empty set is *finite with cardinality* 0. Sets that do not have finite cardinality are *infinite sets* A set S is *countable* if it has the same cardinality as some subset of $\mathbb{N}$. Set that are not countable are said to be *uncountable*.

Note. Saying set S is countable is equivalent to saying that there is a one-to-one (maybe not onto) function from S to $\mathbb{N}$. The next theorem is out first result on cardinalities.

Theorem 1-19. The union of a countable collection of countable sets is countable.

Note. We can represent Theorem 1-19 graphically as follows:



Note 1.3.D. The idea given above can be used to show that $\mathbb{Q}$ is rational. We can create a "list" of all the positive rationals along an infinite array using different values of numerators and denominators. We can then snake through this array and produce a list of the positive rationals (more accurately, we can produce a one-to-one function from the rationals to $\mathbb{N}$). In the production of this list, we need to skip some of the elements in the array since they repeat the same rational number (eg., $1/1 = 2/2 = 3/3 = \cdots$). Similarly, the negative rationals are equal and by Theorem 1-19 the total set $\mathbb{Q}$ is countable. The array looks like this:

1/1	1/2	1/3	1/4	1/5	⋯
2/1	2/2	2/3	2/4	2/5	⋯
3/1	3/2	3/3	3/4	3/5	⋯
4/1	4/2	4/3	4/4	4/5	⋯
5/1	5/2	5/3	6/4	5/5	⋯
⋮	⋮	⋮	⋮	⋮	⋱

Note. The next result is very surprising. It was presented by Georg Cantor in "Ueber eine elementare Frage der Mannigfaltigkeitslehre" ["On an elementary question of the theory of diversity"], *Jahresbericht der Deutschen Mathematiker-Vereinigung*, **1**, 75–78 (1891), which can be viewed online (in German, of course; accessed 11/24/2023). It shows that some infinite sets are larger than others (in the sense of cardinality). In particular, the set of rational numbers is "smaller" than the set $\{x \in \mathbb{R} \mid 0 < x < 1\} = (0, 1)$. The proof technique used is called the *Cantor diagonalization argument*.

Theorem 1-20. The real numbers in $(0, 1)$ form an uncountable set.

Note. To further address cardinality, we introduce one more operation on a set.

Definition. The *power set* of a set X, denoted $\mathcal{P}(X)$, is the set of all subsets of X.

Note. If X is of cardinality n, then $\mathcal{P}(X)$ is of cardinality 2^n. This can be shown with an easy inductive proof, as is to be done in Exercise 1.3.A.

Definition. A *cardinal number* is associated with a set. Two sets share the same cardinal number if they are of the same cardinality. The cardinal number of set X is denoted $|X|$. We order the cardinal numbers with the following:

(i) If X and Y are sets and there is a one to one function from X into Y, then the cardinal number of X is no larger than the cardinal number of Y, denoted $|X| \leq |Y|$ or $|Y| \geq |X|$.

(ii) If (i) holds, and if there is no onto function from X to Y, then the cardinal number of Y is strictly larger than the cardinal number of X, denoted $|X| < |Y|$ or $|Y| > |X|$.

Note. We interpret the idea of the cardinal number of Y being strictly larger than the cardinal number of X as meaning that Y has "more" elements than X. Now that we have seen that some infinite sets are larger than others, we might wonder if there is a "largest" set. The next result show that there is not.

Theorem 1-21. Cantor's Theorem.
The cardinal number of $\mathcal{P}(X)$ is strictly larger than the number of X.

Note. Georg Cantor presented a survey of cardinal numbers (and "transfinite arithmetic") in two papers:

1. Beiträge zur Begründung der transfiniten Mengenlehre" ["Contributions to the Founding of the Theory of Transfinite Numbers"], *Mathematische Annalen*, **46**, 481–512 (1895).

2. Beiträge zur Begründung der transfiniten Mengenlehre" ["Contributions to the Founding of the Theory of Transfinite Numbers"], *Mathematische Annalen*, **49**, 207–246 (1897).

These are in print today in English as: Georg Cantor, *Contributions to the Founding* *of the Theory of Transfinite Numbers*, Translated and Provided with an Introduction and Notes by Philip E. B. Jourdain, Dover Publications (1955). It is also available online in PDF on Andrew Ranicki's webpage (accessed 11/24/2023).

Exercise 1.3.9. Let $y > 0$, $n \in \mathbb{N}$, and $A = \{x \mid x^n < y\}$.

(a) A is nonempty and bounded above.

(b) If $a \in \mathbb{R}$, $a > 0$, and $n \in \mathbb{N}$, then there exists $x \in \mathbb{R}$, $x > 0$, such that $x^n < a$. Use this to prove Theorem 1-8.

Exercise 1.3.10. Let A be uncountable and B countable. Then $A \setminus B$ is uncountable.

Note 1.3.E. Georg Ferdinand Cantor was born in in Saint Petersburg, Russia in 1845. At age 11 his family moved to Germany. He attended the University of Berlin (he was friends with Hermann Schwarz, of the Schwarz Inequality or the Cauchy-Schwarz Inequality from linear algebra). He attended lectures of Karl Weierstrass and Leopold Kronecker. He finished his dissertation on number theory in 1867. At the encouragement of his colleague, Eduard Heine (of the Heine-Borel Theorem fame) his research turned to analysis. He worked on trig series, and defined irrational numbers in terms of convergent sequences of rational numbers. This definition was referenced by Richard Dedekind in his successful development of the idea of completeness of the real numbers in terms of Dedekind cuts. In

1873, Cantor proved that the rational numbers are countable. He also showed the algebraic numbers (that is, real numbers that are roots of polynomials with integer coefficients) are countable. In December 1973 he proved that the real numbers are *not* countable and published this in 1874 (this is where the idea of a one-to-one correspondence enters in relation to cardinality). He further elaborated on these ideas in an 1878 paper. Between 1879 and 1884 he published six papers on set theory. Cantor's theory of sets was not accepted as widely as he had hoped and it was drawing criticism. In May 1884 Cantor had an attack of depression that lasted a few weeks. It was speculated that his depression was the result of the criticism that he suffered through, but in light of modern ideas of mental illness this is not thought to be a major contributor to his problems. Cantor's last major set theory papers were published in 1895 and 1897 (he had been working on the Continuum Hypothesis, the claim that there is no set of cardinality greater than the cardinality of the natural numbers and less than the cardinality of the real numbers, and he wanted to include a proof of this in his papers, but he could not find one; this claim turn out to be neither true nor false under the accepted axioms of set theory. . .). Following this time and some personal tragedies, Cantor continued to suffer depression. He took a leave from teaching in the winter semester of 1899-1900, and spent time off-and-on in a sanatorium from 1899 onwards. He formally retired in 1913. He entered a sanatorium in 1917 and died of a heart attack January 6, 1918. This biographical information and the following image are from the MacTutor History of Mathematics Archive biography of Cantor (accessed 11/24/2023).

Georg Cantor

March 3, 1845–January 6, 1918

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