Chapter 2. Sequences of Real Numbers

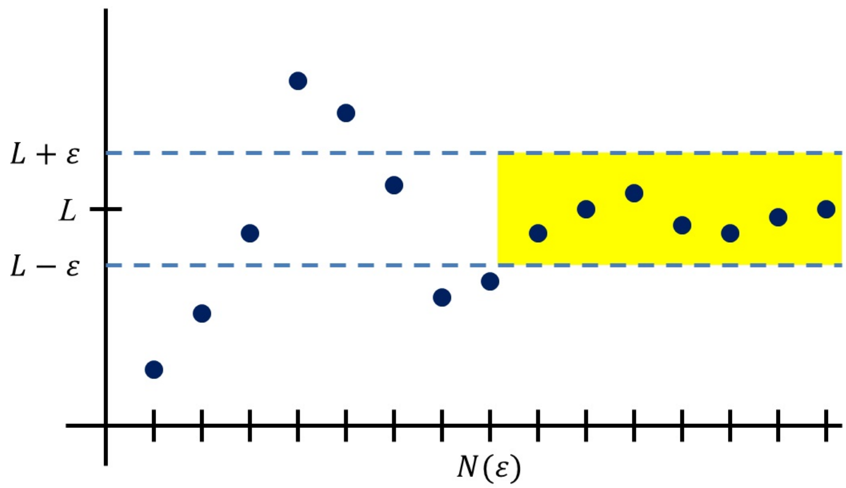
2.1. Sequences of Real Numbers

Note. In this section we define a sequence of real numbers and the limit of a sequence. We prove several properties sequences. We start with the definition of a sequence and the definition of the limit of a sequence, which are the same as those seen in Calculus 2 (MATH 1920); see my online notes for Calculus 2 on Section 10.1. Sequences.

Definition. A *sequence* of real numbers is a function from $\mathbb{N}$ into $\mathbb{R}$. We denote the sequence as $\{f(1), f(2), \ldots\}$ (notice that order matters) $\{f(n)\}_{n=1}^{\infty} = \{f(n)\} = \{f_n\}$ or "$f(1), f(2), \ldots$." The numbers $f(1), f(2), f(3), \ldots$ are the *terms* of the sequence, with $f(n)$ as the nth term. Two sequences $\{f_n\}$ and $\{g_n\}$ are *equal* if they are equal term by term (that is, $f(n) = f_n = g_n = g(n)$ for all $n \in \mathbb{N}$).

Example 2.2. The sequence $2, 4, 6, 8, \ldots$ or $\{2, 4, 6, 8, \ldots\}$ may be represented as $\{2n\} = \{2n\}_{n=1}^{\infty}$. Similarly, when $f(n)$ is easily expressed as a formula, we may represent the sequence by the formula.

Definition. We say that $\{x_n\}_{n=1}^{\infty}$ *converges to* L if for all $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that for all $n > N(\varepsilon)$, we have $|x_n - L| < \varepsilon$. L is called the *limit* of the sequence and we write $\lim(x_n) = L$ or $\{x_n\} \to L$. If a sequence does not converge, it *diverges*.



Note 2.1.A. The idea behind this definition is that you start with a positive distance as small as you like, ε. Then I can find a natural number, $N(\varepsilon)$, such that when we go out the sequence beyond the $N(\varepsilon)$ position, the terms will be within the distance you chose of the limit value L. That is, for any given small positive distance ε, the terms of the sequence are *eventually* (beyond $N(\varepsilon)$ in the sequence) within that distance of the limit value L. Notice that there is no idea of the terms getting "closer and closer" to L. Limits are more subtle than this. An idea of closeness (in the form of $\varepsilon > 0$) is involved, but not an idea of closer and closer. Another informal ideas is that the terms of the sequence "get close to" L (within $\varepsilon > 0$) and stay close of L (for all $n > N(\varepsilon)$.

Note 2.1.B. By the way, Kirkwood requires that $N(\varepsilon)$ is a natural number, but we could replace this requirement with: "$N(\varepsilon)$ is a positive real number." After all, if $N(\varepsilon)$ is a positive real number then we can round it up to $\lceil N(\varepsilon) \rceil$ to get a natural number.

Example 2.1.A. Prove $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \to 0$.

Example 2.4. Prove that $\{x_n\} = \{2 - 1/n^2\}$ has a limit of 2.

Note. Recall that a sequence is *divergent* if it is not convergent. We now consider two special types of divergence.

Definition. A sequence of real numbers $\{x_n\}$ is said to *diverge to infinity* is given any number M, there exists $N(M) \in \mathbb{N}$ such that for all $n > N(M)$ we have $x_n > M$. We write $\lim x_n = \infty$ or $\{x_n\} \to \infty$. A sequence of real numbers $\{x_n\}$ is said to *diverge to negative infinity* is given any number K, there exists $N(K) \in \mathbb{N}$ such that for all $n > N(K)$ we have $x_n < K$. We write $\lim x_n = \infty$ or $\{x_n\} \to -\infty$.

Example 2.6. Prove that $\{x_n\} = \{n^2\}$ diverges to ∞.

Theorem 2-1. A sequence of real numbers can converge to at most one number.

Theorem 2-2. The sequence of real numbers $\{a_n\}$ converges to L if and only if for all $\varepsilon > 0$, all but a finite number of terms of $\{a_n\}$ lie in $(L - \varepsilon, L + \varepsilon)$.

Definition. A sequence is *bounded* if the terms of the sequence form a bounded set.

Note. Boundedness of a sequence will be a recurring theme in our study of sequences. We start by showing that convergent sequences are bounded.

Theorem 2-3. If $\{a_n\}$ is a convergent sequence of real numbers, then the sequence $\{a_n\}$ is bounded.

Definition. If $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers, define $\{a_n + b_n\}$ to be the sequence whose nth term is $a_n + b_n$, define $\{a_n b_n\}$ as the sequence whose nth term is $a_b n_n$ and if $b_n \neq 0$ for all $n \in \mathbb{N}$, define $\{a_n/b_n\}$ as the sequence whose nth term is a_n/b_n. For $c \in \mathbb{R}$, define $c\{a_n\} = \{ca_n\}$ as the sequence whose nth term is ca_n.

Theorem 2-4. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences with $\{a_n\} \to a$ and $\{b_n\} \to b$. Then

(a) $\{a_n + b_n\} \to a + b$.

(b) $\{ca_n\} \to ca$ for any $c \in \mathbb{R}$.

(c) $\{a_n b_n\} \to ab$.

(d) If $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then $\{a_n/b_n\} \to a/b$.

Note. The proof of part (b) of Theorem 2-4 is to be given in Exercise 2.1.7. The proof of the next theorem is to be given in Exercises 2.1.8 and 2.1.9. In the proof of

Theorem 2-4(c), we established the inequality $|a_n b_n - ab| \le |a_n||b_n - b| + |b||a_n - a|$. We needed to chose $N(\varepsilon)$ such that the quantity on the right is less than ε. The plan is to make each of the two quantities in the sum less than $\varepsilon/2$. The second term involves $|a_n - a|$ and, since $\{a_n\} \to a$ then this can be made "small" by making n "large." So we choose $N_a(\varepsilon)$ such that for $n > N_a(\varepsilon)$ we have $|a_n - a| < \varepsilon/(2|b|+1)$ (the "+1" is included in case $b = 0$). Then

$$|b||a_n - a| < |b|\left(\frac{\varepsilon}{2|b|+1}\right) = \left(\frac{|b|}{2|b|+1}\right)\varepsilon < \frac{\varepsilon}{2}.$$

The first term involves $|b_n - b|$ and, since $\{b_n\} \to b$ then this can be made "small" by making n "large. The problem is that the variable term $|a_n|$ is in the way, so we need some idea of its size. Since $|a_n|$ is convergent, then it is bounded, say by $M > 0$. So we choose $N_b(\varepsilon)$ such that for $n > N_b(\varepsilon)$ we have $|b_n - b| < \varepsilon/(2M)$. Then

$$|a_n||b - b_n| < M\left(\frac{\varepsilon}{2M}\right) = \frac{\varepsilon}{2}.$$

To get both of these to hold, we choose $N = \max\{N_a(\varepsilon), M_b(\varepsilon)\}$. A similar approach is taken in the proof of Theorem 2-4(d) where we deal with the inequality

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| \le \frac{1}{|b_n|}|a_n - a| + \frac{|a|}{|b_n b|}|b - b_n|.$$

The details are given in Kirkwood as to the specific choices of $N_a(\varepsilon)$ and $N_b(\varepsilon)$.

Theorem 2-5.

(a) Suppose $\{a_n\} \to L$ and $a_n \le K$ for all $n \in \mathbb{N}$. Then $L \le K$.

(b) Suppose $\{a_n\}$, $\{b_n\}$ satisfy $a_n \le b_n$ for all $n \in \mathbb{N}$. Also suppose $\{a_n\} \to L$, $\{b_n\} \to K$. Then $L \le K$.

(c) If $\{a_n\}$, $\{b_n\}$ satisfy $0 \le a_n \le b_n$ for all $n \in \mathbb{N}$ and if $\{b_n\} \to 0$, then $\{a_n\} \to 0$.

(d) If $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ satisfy $a_n \le b_n \le c_n$ for all $n \in \mathbb{N}$, and $\{a_n\} \to L$, $\{c_n\} \to L$, then $\{b_n\} \to L$.

Definition. Let $\{a_n\}$ be a sequence. If $a_{n+1} \ge a_n$ $(a_n \ge a_{n+1})$ for all $n \in \mathbb{N}$, the sequence is *monotone increasing* (*decreasing*). If the inequalities are strict for all $n \in \mathbb{N}$, then the sequence is *strictly monotone increasing* (*decreasing*). Any of these types of sequences is called a *monotone sequence*.

Theorem 2-6. A bounded monotone sequence converges.

Corollary 2-6.

(a) A monotone increasing sequence either converges or diverges to ∞.

(b) A monotone decreasing sequence either converges or diverges to $-\infty$.

Example 2.9. Prove that the sequence $\{x_n\} = \{(1 + 1/n)^n\}$ is monotone increasing.

Theorem 2-7. Let $A_n = [a_n, b_n]$ be a sequence of nested intervals, $A_n \supset A_{n+1}$ for all $n \in \mathbb{N}$. Suppose $\lim_{n\to\infty}(b_n - a_n) = 0$. Then there exists $p \in \mathbb{R}$ such that $\cap_{n=1}^{\infty} A_n = \{p\}$.

Note/Defintion. A proof of Theorem 2-7 is to be given in Exercise 2.1.12. A sequence of sets $\{A_n\}$ such that $A_n \supset A_{n+1}$ is a *nested* sequence of sets.

Theorem 2-8. Let A be a nonempty set of real numbers bounded above. Then there is a sequence $\{x_n\}$ such that (i) $x_n \in A$ for all $n \in \mathbb{N}$, and (ii) $\{x_n\} \to \text{lub}(A)$.

Note. A result similar to Theorem 2-8 also holds for the greatest lower bound of a set bounded below:

> Let A be a nonempty set of real numbers bounded below. Then there is a sequence $\{x_n\}$ such that (i) $x_n \in A$ for all $n \in \mathbb{N}$, and (ii) $\{x_n\} \to \text{glb}(A)$.

Definition. A sequence $\{a_n\}$ is a *Cauchy sequence* if for all $\varepsilon > 0$, there exists $N(\varepsilon)$ such that if $n, m > N(\varepsilon)$ then $|a_n - a_m| < \varepsilon$.

Note 2.1.C. Cauchy sequences will play a huge role in this chapter. The theoretical importance of Cauchy sequences is that they make no appeal to anything outside of the sequence, but merely address a property of the terms of the sequence itself. Notice that in the definition of convergence, by contrast, the role of the limit of the sequence may or may not be an element of the sequence. We'll apply Cauchy sequences of rational numbers when showing the uniqueness a complete ordered field in Supplement. The Real Numbers are the Unique Complete Ordered Field. In fact, completeness is usually addressed in terms of Cauchy sequences when an ordering is not present (and there is no concept of least/greatest or lower/upper). This is the case in complex analysis (see my online notes for Complex Analysis 1 [MATH 5510] on Section I.3. The Complex Plane and notice Note 1.3.D), and in metric spaces (see my online notes for Introduction to Topology [MATH 4357/5357] on Section 43. Complete Metric Spaces, and Real Analysis 2 [MATH 5220] on Section 9.4. Complete Metric Spaces). In Exercises 2.3.13 and 2.3.14 of Section 2.3. Bolzano-Weierstrass Theorem a proof of the following is to be given.

Theorem 2-9. A sequence converges if and only if it is Cauchy.

Note 2.1.D. The ideas of continuity and a continuum are old. The first attempt at a formal definition of continuity was given in 1817 by Bernhard Bolzano (October 5, 1781–December 18, 1848) in a pamphlet in which he was trying to prove the Intermediate Value Theorem; the pamphlet was in German and the title translates as *Purely analytic proof of the theorem that between any two values which give results of opposite sign, there lies at least one real root of the equation*). In this, Bolzano was critical of the "common sense" approaches to continuity of the time. In Sections 6 and 7 of the pamphlet he tries to prove that Cauchy sequences (which he describes independently from Cauchy) converge. (He is actually considering a series, but the convergence of a series is addressed in terms of convergence of the sequence of partial sums.) He introduces a new assumption of the existence of a quantity X to which the terms of the sequence approach as closely as wanted. He claimed this hypothesis "contains nothing impossible," but it is what he was trying to prove. Without some new hypothesis (namely, completeness in one form or another), the proof could not be correct. Augustin Cauchy (August 21, 1789–May 23, 1857) published his *Cours d'analyse* in 1821. It was for the students at École Polytechnique and was an attempt to put calculus on a rigorous foundation. In addressing series (and the associated sequences of partial sums) he assumed that "Cauchy sequences" converge and stated that the property of being a Cauchy sequence was "a self evident necessary and sufficient condition" for convergence. In other words, he *assumed* these sequences converged! In fact, the convergence of Cauchy sequences in an ordered field (along with the Archimedean Principle, Theorem 1-18) is equivalent to the Axiom of Convergence we used in terms of upper bounds and least upper bounds; this is shown in Theorem 1.4.3 of Supplement. The Real Numbers are the Unique Complete Ordered Field. This note is based on Jacquleine Stedall's *Mathematics Emerging: A Sourcebook 1540–1900* (Oxford University Press, 2008), pages 306, 495, 496, and 500. This also includes Sections 1, 3, and 4 of *Cours d'analyse* (in French) and an English translation. It would be Richard Dedekind (October 6, 1831–February 12, 1916) who introduced the idea of a "Dedekind cut" (in 1858, but not published until 1872) as an axiom and it is from this point that the real numbers were clearly defined axiomatically. Dedekind cuts can be used to prove that Cauchy sequences converge and that sets of real numbers with an upper bound have a least upper bound. That is, Dedekind cuts can be used to establish the completeness of the real numbers (and conversely).

These images of Bernhard Bolzano and Augustin Cauchy are from the Bolzano biography webpage and the Cauchy biography webpage of the MacTutor History of Math website (accessed 11/26/2023).

Revised: 2/17/2024