Chapter 2. Sequences of Real Numbers.

2.1. Sequences of Real Numbers.

Note. In this section we define a sequence of real numbers and the limit of a sequence (topics which you have seen in Calculus 2 [MATH 1920]).

Definition. A sequence of real numbers is a function from \( \mathbb{N} \) into \( \mathbb{R} \). We denote the sequence as \( \{f(1), f(2), \ldots\} \) (notice that order matters) \( \{f(n)\}_{n=1}^{\infty} \) or “\( f(1), f(2), \ldots \)”

Definition. We say that \( \{x_n\}_{n=1}^{\infty} \) converges to \( L \) if for all \( \varepsilon > 0 \) there exists \( N(\varepsilon) \) such that for all \( n > N(\varepsilon) \), we have \( |x_n - L| < \varepsilon \). \( L \) is called the limit of the sequence and we write \( \lim(x_n) = L \) or \( \{x_n\} \rightarrow L \). If a sequence does not converge, it diverges.
Example. Prove $\left\{ \frac{1}{n} \right\}_{n=1}^{\infty} \rightarrow 0$.

Definition. A sequence of real numbers $\{x_n\}$ is said to diverge to infinity is given any number $M$, there exists $N(M) \in \mathbb{N}$ such that

$$\text{for all } n > N(M) \text{ we have } x_n > M.$$ 

We write $\lim x_n = \infty$ or $\{x_n\} \rightarrow \infty$.

Note. We can define diverge to negative infinity similarly.

Theorem 2-1. A sequence of real numbers can converge to at most one number.

Theorem 2-2. The sequence of real numbers $\{a_n\}$ converges to $L$ if and only if for all $\varepsilon > 0$, all but a finite number of terms of $\{a_n\}$ lie in $(L - \varepsilon, L + \varepsilon)$.

Definition. A sequence is bounded if the terms of the sequence form a bounded set.

Theorem 2-3. If $\{a_n\}$ is a convergent sequence of real numbers, then the sequence $\{a_n\}$ is bounded.
Definition. If \{a_n\} and \{b_n\} are sequences of real numbers, define \{a_n + b_n\} to be the sequence whose \(n\)th term is \(a_n + b_n\), define \{a_nb_n\} as the sequence whose \(n\)th term is \(a_nb_n\) and if \(b_n \neq 0\) for all \(n \in \mathbb{N}\), define \{\(a_n/b_n\)\} as the sequence whose \(n\)th term is \(a_n/b_n\). For \(c \in \mathbb{R}\), define \(c\{a_n\} = \{ca_n\}\) as the sequence whose \(n\)th term is \(ca_n\).

Theorem 2-4. Suppose \{a_n\} and \{b_n\} are sequences with \(\{z_n\} \rightarrow a\) and \(\{b_n\} \rightarrow b\). Then

(a) \(\{a_n + b_n\} \rightarrow a + b\).

(b) \(\{ca_n\} \rightarrow ca\) for any \(c \in \mathbb{R}\).

(c) \(\{a_nb_n\} \rightarrow ab\).

(d) If \(b \neq 0\) and \(b_n \neq 0\) for all \(n \in \mathbb{N}\), then \(\{a_n/b_n\} \rightarrow a/b\).

Theorem 2-5.

(a) Suppose \(\{a_n\} \rightarrow L\) and \(a_n \leq K\) for all \(n \in \mathbb{N}\). Then \(L \leq K\).

(b) Suppose \(\{a_n\}\), \(\{b_n\}\) satisfy \(a_n \leq b_n\) for all \(n \in \mathbb{N}\). Also suppose \(\{a_n\} \rightarrow L\), \(\{b_n\} \rightarrow K\). Then \(L \leq K\).

(c) If \(\{a_n\}\), \(\{b_n\}\) satisfy \(0 \leq a_n \leq b_n\) for all \(n \in \mathbb{N}\) and if \(\{b_n\} \rightarrow 0\), then \(\{a_n\} \rightarrow 0\).

(d) If \(\{a_n\}\), \(\{b_n\}\), \(\{c_n\}\) satisfy \(a_n \leq b_n \leq c_n\) for all \(n \in \mathbb{N}\), and \(\{a_n\} \rightarrow L\), \(\{c_n\} \rightarrow L\), then \(\{b_n\} \rightarrow L\).
Definition. Let \( \{a_n\} \) be a sequence. If \( a_{n+1} \geq a_n \) (\( a_n \geq a_{n+1} \)) for all \( n \in \mathbb{N} \), the sequence is monotone increasing (decreasing). If the inequalities are strict for all \( n \in \mathbb{N} \), then the sequence is strictly monotone increasing (decreasing).

**Theorem 2-6.** A bounded monotone sequence converges.

**Corollary 2-6.**

(a) A monotone increasing sequence either converges or diverges to \( \infty \).

(b) A monotone decreasing sequence either converges or diverges to \( -\infty \).

Note. Example 2-9 shows that \( \{(1 + 1/n)^n\} \) is monotone increasing.

Definition. A sequence of sets \( \{A_n\} \) such that \( A_n \supset A_{n+1} \) is a nested sequence of sets.

**Theorem 2-7.** Let \( A_n = [a_n, b_n] \) be a sequence of nested intervals, \( A_n \supset A_{n+1} \) for all \( n \in \mathbb{N} \). Suppose \( \lim_{n \to \infty} (b_n - a_n) = 0 \). Then there exists \( p \in \mathbb{R} \) such that \( \bigcap_{n=1}^{\infty} A_n = \{p\} \).

**Theorem 2-8.** Let \( A \) be a nonempty set of real numbers bounded above. Then there is a sequence \( \{x_n\} \) such that (i) \( x_n \in A \) for all \( n \in \mathbb{N} \), and (ii) \( \{x_n\} \to \text{lub}(A) \).
Definition. A sequence \( \{a_n\} \) is a Cauchy sequence if

for all \( \varepsilon > 0 \), there exists \( N(\varepsilon) \) such that

if \( n, m > N(\varepsilon) \) then \( |a_n - a_m| < \varepsilon \).

Theorem 2-9. A sequence converges if and only if it is Cauchy.

Note. A proof of Theorem 2-9 is given in Exercises 2.3.13 and 2.3.14.

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