2.2. Subsequences.

Note. In this section we introduce subsequences of a given sequence and see how they relate to the original given sequence. In Section 2.3. Bolzano-Weierstrass Theorem we will define the "limit supremum" and "limit infimum" of a sequence, and relate these to subsequences (in Exercise 2.3.16).

Definition. Let $n_1 < n_2 < \cdots < n_k < \cdots$ be strictly increasing sequence of positive integers. Then $a_{n_1}, a_{n_2}, \ldots, a_{n_k}, \ldots$ is a *subsequence* of $\{a_n\}$ and is denoted $\{a_{n_k}\}$.

Note 2.2.A. In subsequence $\{a_{n_k}\}$ of sequence $\{a_n\}$, the entry in the kth position is a_{n_k} so that the position in the subsequence is given by the second subscript. A subsequence of a given sequence is simply a subset of the set of elements of the given sequence listed in the same relative order. Notice that $k \le n_k$. Convergence and the limit of a subsequence is defined as the same way as the convergence and limit of a sequence as given in Section 2.1. Sequences of Real Numbers, with the role of n in the sequence replaced with k for the subsequence. For sequence $\{-1, 1, -1, 1, -1, 1, \ldots\} = \{(-1)^n\} = \{a_n\}$, an example of a subsequence is $\{1, 1, 1, 1, \ldots\} = \{(-1)^{2n}\} = \{a_{n_k}\}$. Here, we have $n_k = 2k$. Notice that $\{a_n\}$ diverges, yet $\{a_{n_k}\}$ converges to 1.

Definition. L is a *subsequential limit* of sequence $\{a_n\}$ if there is a subsequence of $\{a_n\}$ that converges to L.

Note 2.2.B. We consider **by convention** that ∞ or $-\infty$ are subsequential limits of a sequence if there are subsequences that diverge to ∞ or $-\infty$, respectively. Also **by convention** we take as the least upper bound of a set that is not bounded above as ∞, and the greatest lower bound of a set that is not bounded below as $-\infty$. With these conventions, every sequence has a subsequential limit and every set has a least upper bound and a greatest lower bound. Another **convention** is that we use a letter (such as "L") to indicate a limit or subsequential limit that is a real number (as opposed to ∞ or $-\infty$). Kirkwood also refers to a subsequential limit of a "limit point" of a sequence. However, these notes avoid this last bit of terminology; in these notes we use the terminology of "limit" in the setting of sequences or subsequences, and the term "limit point" in the setting of sets (in Section 2.3. Bolzano-Weierstrass Theorem).

Theorem 2-10. A sequence $\{a_n\}$ converges to L if and only if every subsequence of $\{a_n\}$ converges to L.

Note. Now we give an ε classification of subsequential limits.

Theorem 2-11. Real number L is a subsequential limit of $\{a_n\}$ if and only if $\varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains infinitely many terms of $\{a_n\}$.

Exercise 2.2.8(a). Construct a sequence with exactly two subsequential limits. Can this be done is such a way that no two terms of the sequence are the same?

Note. A corollary to Theorem 2-11 also gives an ε classification of subsequential limits.

Corollary 2-11. Let $\{a_n\}$ be a sequence. Then L is a subsequential limit of $\{a_n\}$ if and only if for all $\varepsilon > 0$ and for all $N \in \mathbb{N}$, there exists $n(\varepsilon, N) \in \mathbb{N}$, $n(\varepsilon, N) > N$, such that $|a_{n(\varepsilon,N)} - L| < \varepsilon$.

Exercise 2.2.12(a). If $\{a_n\} \to L$ and if $a_n \leq L$ for infinitely many values of n, then there is a subsequence of $\{a_n\}$ that is increasing (i.e., nondecreasing) and converges to L.

Note 2.2.C. We can use Theorem 2-11 to create sequences with lots of subsequential limits. Consider the following back-and-forth dialogue:

Question 1. Can you find a sequence with every natural number as a subsequential limit?

Answer. YES! Consider $\{1; 1, 2; 1, 2, 3; 1, 2, 3, 4; \ldots\}$.

Question 2. Can you find a sequence with every rational number as a subsequential limit?

Answer. YES! Let $\{q_n\}$ be an enumeration of the rationals and consider $\{q_1; q_1, q_2;\ q_1, q_2, q_3; q_1, q_2, q_3, q_4; \ldots\}$.

Question 3. Can you find a sequence with every real number as a subsequential limit?

Answer. YES! Take the sequence $\{q_n\}$ as above and use an ε argument.

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