2.3. Bolzano-Weierstrass Theorem

Note. In this section we show that every bounded set of real numbers has a "limit point" in the Bolzano-Weierstrass Theorem (Theorem 2-12). We define "limit superior" and "limit inferior" of a sequence, and relate these to limit points of the set of terms of the sequence. We also give a proof of Theorem 2-9 from Section 2.1. Sequences of Real Numbers, which claims that a sequence of real numbers is Cauchy if and only if it converges.

Definition. A real number x is a *limit point* of a set of real numbers A is for all $\varepsilon > 0$, the interval $(x - \varepsilon, x + \varepsilon)$ contains infinitely many points of A.

Note. Of course, a finite set has no limit points. An infinite set may not have a limit point; consider for example $\mathbb{Z}$. Notice that $A = (0, 1)$ has every one of its elements as a limit point, as well as limit points 0 and 1. The Bolzano-Weierstrass Theorem gives a condition under which a set must have at least one limit point.

Theorem 2-12. Bolzano-Weierstrass Theorem.
Every bounded infinite set of real numbers has at least one limit point.

Note 2.3.A. We mentioned Bernhard Bolzano (October 5, 1781–December 18, 1848) in Section 2.1. Sequences of Real Numbers in connection with the convergence of Cauchy sequences (see Note 2.1.D). In E. T. Bell's *Men of Mathematics* (Simon and Schuster, 1937) (see my History of Mathematics [MATH 3040] on Section 3.2. Pythagoras and the Pythagoreans for some commentary on *Men of Mathematics*, including the "non-modern" title; see Note 3.2.A), Karl Wilhelm Weierstrass (October 31, 1815–February 19, 1897) is described as "conclusive" evidence that someone devoted to teaching can still make accomplishments in mathematics. In Berlin, in the process of lecturing on introductory analysis (presumably, covering material very similar to the content of *our* class) in 1859–60, he addressed the foundations of analysis for the first time. He followed this up in 1860–61 with lectures on integration theory. In 1863–64 he taught a course on the general theory of analytic functions and began to formulate his theory of the real numbers. As part of his rigorous approach, he defined irrational numbers as limits of convergent series. We'll see a similar approach taken in Supplement. The Real Numbers are the Unique Complete Ordered Field. Though he did not publish a lot, Weierstrass is often called the "father of modern analysis."

This biographical information and the image above are from the MacTutor biography webpage on Weierstrass (accessed 12/1/2023).

Note. The next result, a classification of subsequential limits in terms of limit limit points of the set of terms of the sequence, is an application of the Bolzano-Weierstrass Theorem (Theorem 2-12). Its proof is to be given in Exercise 2.3.1.

Theorem 2-13. Let $\{a_n\}$ be a sequence. Then L is a (finite) subsequential limit of $\{a_n\}$ if and only if L satisfies either of the following:

(i) There are infinitely many terms of $\{a_n\}$ equal to L, or

(ii) L is a limit point of a set consisting of the terms of $\{a_n\}$.

Note. The next result gives a sufficient condition which gives a condition for a sequence to have a convergent subsequence (though the condition is not necessary).

Theorem 2-14. Every bounded sequence has a convergent subsequence.

Note. The next result suggests that we can include $+\infty$ and $-\infty$ as subsequential limits (though often when we refer to a "subsequential limit" we are referring to the limit of a subsequence, and subsequences that have limits of $+\infty$ or $-\infty$ are said to be *divergent*). See Note 2.2.B of Section 2.2. Subsequences for more on this. A proof of part (b) of the next result is to be given in Exercise 2.3.4.

Theorem 2-15.

(a) A sequence that is unbounded above has a subsequence that diverges to $+\infty$.

(b) A sequence that is unbounded below has a subsequence that diverges to $-\infty$.

Note. The following theorem gives another classification of convergent sequences.

Theorem 2-16. A sequence $\{a_n\}$ converges if and only if it is bounded and has exactly one subsequential limit.

Note. We introduce one final parameter related to subsequences of a given sequence. You will see a lot on this parameter in your study of analysis. For example, it is used to define the radius of convergence of a power series in Section 8.2. Series of Functions. This is also the case in the complex setting; see my online notes for Complex Analysis 1 (MATH 5510) on Section III.1. Power Series (notice Theorem III.1.3).

Definition. Let $\{a_n\}$ be a sequence of real numbers. Then $\limsup a_n = \overline{\lim}\, a_n$ is the least upper bound of the set of subsequential limits of $\{a_n\}$, and $\liminf a_n = \underline{\lim}\, a_n$ is the greatest lower bound of the set of subsequential limits of $\{a_n\}$.

Note 2.3.B. By using least upper bounds and greatest lower bounds in the previous definition, we are guaranteed that $\limsup a_n = \overline{\lim}\, a_n$ and $\liminf\, a_n = \underline{\lim}\, a_n$ exist for all sequences of real numbers $\{a_n\}$. Of course, these might be $-\infty$ or $+\infty$, however. In Complex Analysis 1 (MATH 5510), for sequence $\{a_n\}$ of real numbers, $\limsup a_n = \overline{\lim}\, a_n$ and $\liminf\, a_n = \underline{\lim}\, a_n$ are defined as:

$$\underline{\lim}\, a_n = \lim_{n\to\infty} \left(\inf\{a_n, a_{n+1}, \ldots\}\right) \text{ and } \overline{\lim}\, a_n = \lim_{n\to\infty} \left(\sup\{a_n, a_{n+1}, \ldots\}\right).$$

This is equivalent to our definition and explains the terminology "$\liminf a_n$" and "$\limsup a_n$."

Example 2.3.A. Consider $\{a_n\} = \{\sin n\}$ (n in radians). Then $\underline{\lim}\, a_n = -1$ and $\overline{\lim}\, a_n = 1$. The proof of this claim is not trivial. A recent graduate of the ETSU Mathematical Sciences Master's Program, Abderrahim Elallam, presented a proof of this in his thesis *Constructions & Optimization in Classical Real Analysis Theorems* (May 2021). In his Section 2.3, "Constructions in the Bolzano-Weierstrass Theorem," he proved:

Proposition 2.1. For every $\alpha \in [-1, 1]$ there is a subsequence $\{x_{n_k}\}$ of $\{x_n = n\}$ such that $\lim_{k\to\infty} \sin(x_{n_k}) = \alpha$.

He lists as a reference for this result G. H. Hardy and E. M. Wright's *An Introduction to the Theory of Numbers* (Oxford University Press, 1981). You can see Mr. Elallam's thesis online through the Digital Commons @ East Tennessee State University (accessed 9/18/2023).

Exercise 2.3.16. Let $\{a_n\}$ be a sequence.

(a) Then $\limsup a_n = \overline{\lim}\, a_n$ is a subsequential limit of $\{a_n\}$, and

(b) $\liminf a_n = \underline{\lim}\, a_n$ is a subsequential limit of $\{a_n\}$.

Note 2.3.C. Exercise 2.3.16, gives us an easier way to recognize $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$; they are simply the greatest and least, respectively, subsequential limits (allowing for the possibility of $+\infty$ and $-\infty$). We leave the proof of part (b) of Exercise 2.3.16 ad homework.

Note. The next result gives an ε condition that classifies $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$. A proof of part (b) of the result is to be given in Exercise 2.3.7.

Theorem 2-17. Let $\{a_n\}$ be a bounded sequence. Then

(a) $\overline{\lim}\, a_n = L$ if and only if for all $\varepsilon > 0$, there exists infinitely many terms of $\{a_n\}$ in $(L - \varepsilon, L + \varepsilon)$ but only finitely many terms of $\{a_n\}$ with $a_n > L + \varepsilon$.

(b) $\underline{\lim}\, a_n = K$ if and only if for all $\varepsilon > 0$, there exists infinitely many terms of $\{a_n\}$ in $(K - \varepsilon, K + \varepsilon)$ but only finitely many terms of $\{a_n\}$ with $a_n < K - \varepsilon$.

Note. The next corollary should not be surprising, at this stage.

Corollary 2-17. A bounded sequence $\{a_n\}$ converges if and only if $\overline{\lim}\, a_n = \underline{\lim}\, a_n$.

Note. The familiar summation property of convergent sequence (see Theorem 2-4 in Section 2.1. Sequences of Real Numbers) does not translate directly over to $\overline{\lim}$ and $\underline{\lim}$; an inequality is necessary. Part (b) of the next result, concerning $\overline{\lim}$ and $\underline{\lim}$ of a sum, is to be given in Exercise 2.3.8.

Theorem 2-18.

(a) $\overline{\lim}(a_n + b_n) \leq \overline{\lim}\, a_n + \overline{\lim}\, b_n$, and

(b) $\underline{\lim}\, a_n + \underline{\lim}\, b_n \leq \underline{\lim}(a_n + b_n)$.

Note. Equality does not always hold in Theorem 2-18. Consider $\{a_n\} = \{\sin^2 n\}$ and $\{b_n\} = \{\cos^2 n\}$. Then $\overline{\lim}\, a_n = \overline{\lim}\, b_n = 1$ (this is similar to Example 2.3.A), but $\overline{\lim}(a_n + b_n) = \overline{\lim}(\sin^2 n + \cos^2 n) = 1 < 1 + 1 = 2$.

Definition. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *bounded* if the range of f is a bounded set. For a bounded function denote $\text{lub}(\mathcal{R}(f))$ as $\sup(f)$ and $\text{glb}(\mathcal{R}(f))$ and $\inf(f)$.

Note. The sup and inf of a sum of bounded functions is similar to the behavior of $\overline{\lim}$ and $\underline{\lim}$ of a sum of sequences, as given in Theorem 2-18. The behavior is given in the next result, and part (b) is to be given in Exercise 2.3.A.

Theorem 2-19. Let f and g be bounded functions with the same domain. Then:

(a) $\sup(f + g) \leq \sup(f) + \sup(g)$, and

(b) $\inf(f) + \inf(g) \leq \inf(f + g)$.

Note. Recall that a sequence $\{a_n\}$ is a *Cauchy sequence* if: for all $\varepsilon > 0$, there exists $N(\varepsilon)$ such that if $n, m > N(\varepsilon)$ then $|a_n - a_m| < \varepsilon$. Theorem 2-9 claims that a sequence converges if and only if it is Cauchy. We now give a proof of this claim in the following two exercises verify this claim.

Exercise 2.3.13. Let $\{a_n\}$ be a Cauchy sequence.

(a) Then $\{a_n\}$ is bounded.

(b) There is at least one subsequential limit for $\{a_n\}$.

(c) There is no more than one subsequential limit of $\{a_n\}$.

(d) $\{a_n\}$ converges.

Exercise 2.3.14. A convergent sequence is Cauchy.

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