Chapter 3. Topology of the Real Numbers.

3.1. Topology of the Real Numbers.

Note. In this section we introduce "topological" properties of sets of real numbers such as *open*, *closed*, *compact*, and *connected*. Each of the last three ideas is based on the first one, an open set of real numbers. Open sets are fundamental in the study of every area of analysis. They are vital in defining and using such analytic ideas as limits, continuity, and derivatives. We will classify open sets of real numbers in terms of open intervals in Theorem 3-5; this result is of such colossal importance that we give the proof in a supplement to this section, Supplement: A Classification of Open Sets of Real Numbers. We start with the definition of an open set.

Definition. A set U of real numbers is said to be *open* if for all $x \in U$ there exists $\delta(x) > 0$ such that $(x - \delta(x), x + \delta(x)) \subset U$.

Note 3.1.A. Notice that if U is open and $x \in U$, then there is "wiggle room" around x. That is, we can move a little bit to the left or right of x and stay in set U. The "little bit" we can move is given by the value of $\delta(x)$ in the definition. This ability to wiggle around point x allows us to discuss "closeness." These vague terms are given here to help you think through the associated difficult concepts; everything will be given formally in this class. These ideas are illustrated in the first few days of Calculus 1 (MATH 1910) when you encounter the limit of a function (in our class, we cover this topic in Section 4.1. Limits and Continuity). Recall that in Calculus 1, the ε/δ definition of $\lim_{x\to c} f(x) = L$ requires that f is "defined in an open interval containing c, except possibly c itself. Then it is required that for all $\varepsilon > 0$, there is a $\delta > 0$ such that if $x \in (c-\delta, c+\delta)$, $x \neq c$, then $f(x) \in (L-\varepsilon, L+\varepsilon)$. See my online notes for Calculus 1 on Section 2.3. The Precise Definition of a Limit (the intervals are expressed in terms of inequalities involving absolute values in those notes, but they are equivalent to what is stated here). So, ultimately, all of the ideas of calculus involving limits (which is the vast majority of ideas in calculus) are based on open intervals. Informally, this limit involves "closeness," but it has nothing to do with an idea of "closer and closer," similar to the discussion of limits of sequences in Section 2.1. Sequences of Real Numbers; see Note 2.1.A. Open sets are how "closeness" is defined.

Note. It is trivially true that $\mathbb{R}$ is open. It is vacuously true that $\varnothing$ is open. In the next result it is proved that open intervals are, in fact, open sets. Open intervals are *examples* of open sets, but there are open sets that are not intervals. For example, the union of two disjoint open intervals is also an open set (yet the union is not itself an interval).

Theorem 3-1. The intervals (a, b), (a, ∞), and $(-\infty, a)$ are open sets.

Note. We will classify open sets of real numbers below (in Theorem 3-5 and Supplement. A Classification of Open Sets of Real Numbers). This classification will be in terms of open intervals. Next, we define a "closed set" of real numbers. The definition is somewhat indirect (it is based on a property of the complement of the set). Because of this indirectness, we will only be able to classify a closed set of real numbers indirectly (in fact, in terms of the complement and open intervals). See Note OS.B in Supplement. A Classification of Open Sets of Real Numbers.

Definition. A set A is *closed* if A^c is open.

Note 3.1.B. The sets $\mathbb{R}$ and $\varnothing$ are both closed. In fact, these are the only sets of real numbers that are both open and closed. Some sets are *neither* open *nor* closed.

Corollary 3-1. The intervals $(-\infty, a]$, $[a, b]$, and $[b, \infty)$ are closed sets.

Note. The next result concerns unions and intersections of open sets. We'll see that the union of any collection of open sets is open, but on a finite intersection of open sets is necessarily open. We will see results similar to the next one again (in Theorem 3-4, for example), and it is the motivation for the definition of a *topology* on a set (see Note 3.1.C).

Theorem 3-2. The open sets satisfy:

(a) If $\{U_1, U_2, \ldots, U_n\}$ is a *finite* collection of open sets, then $\cap_{k=1}^n U_k$ is an open set.

(b) If $\{U_\alpha\}$ is *any* collection (finite, infinite, countable, or uncountable) of open sets, then $\cup_\alpha U_\alpha$ is an open set.

Note 3.1.C. Notice that finiteness is required in Theorem 3-2 when considering intersections of open sets. This is because an infinite intersection of open sets can be "not open" (even closed). Consider, for example, $\cap_{i=1}^\infty (-1/i, 1 + 1/i) = [0, 1]$. We should expect some type of analogous behavior for closed sets, since closed sets are simply complements of open sets. The next result gives properties of unions and intersections of closed sets, similar to that of Theorem 3-2 for open sets. The proof of Theorem 3-3, which is to be given in Exercise 3.1.16(a), is based on DeMorgan's Laws (Corollary 1-1 and Exercise 1.1.8). Recall that DeMorgan's Laws deal with complements, unions, and intersections (so it seems the perfectly reasonable thing to apply here, since it involves interchanging closed sets for their complements, namely open sets, and interchanging unions and intersections; this is exactly how Theorem 3-3 relates to Theorem 3-2).

Theorem 3-3. The closed sets satisfy:

(a) $\varnothing$ and $\mathbb{R}$ are closed.

(b) If $\{A_\alpha\}$ is *any* collection of closed sets, then $\cap_\alpha A_\alpha$ is closed.

(c) If $\{A_1, A_2, \ldots, A_n\}$ is a *finite* collection of closed sets, then $\cup_{k=1}^n A_i$ is closed

Note 3.1.D. Just as an infinite intersection of open sets may not be open (as observed in Note 3.1.B), an infinite union of closed sets may not closed. Consider, for example, $\cup_{i=1}^{\infty}[1/i, 1-1/i] = (0,1)$.

Note. We now step aside and briefly discuss topology in a general setting. We are inspired by Theorem 3-2 to give the following definition. A detailed exploration of these ideas are given in Introduction to Topology (MATH 4357/5357); see my online notes for Introduction to Topology.

Definition. A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(a) $\varnothing$ and X are in $\mathcal{T}$.

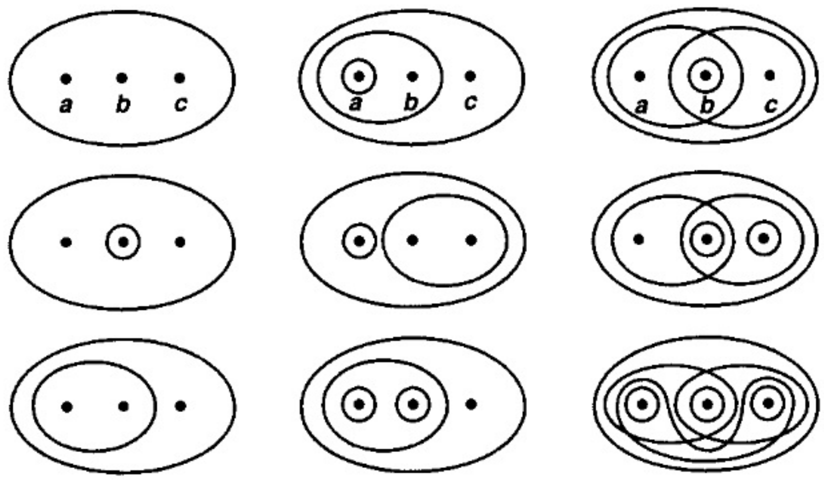
(b) The union of *any* collection of elements of $\mathcal{T}$ is in $\mathcal{T}$.

(c) The intersection of any *finite* collection of elements of $\mathcal{T}$ is in $\mathcal{T}$.

Together, the sets X and $\mathcal{T}$ are called a *topological space*. If $U \in \mathcal{T}$, then U is said to be *open*.

Example 3.1.A. A topology on the real line is given by the collection of intervals of the form (a,b) along with arbitrary unions of such intervals. Let $I = \{(a,b) \mid a,b \in \mathbb{R}\}$. Then the sets $X = \mathbb{R}$ and $\mathcal{T} = \{\cup_\alpha I_\alpha \mid I_\alpha \in I\}$ is a topological space. This is $\mathbb{R}$ under the "usual topology" on $\mathbb{R}$.

Example 3.1.B. Let $X = \{a, b, c\}$. The following are topologies on X. This is Figure 12.1 from page 76 of James Munkres' *Topology: A First Course*, 2nd edition (Prentice Hall, 2000):



Definition. The topology consisting of all subsets of X is called the *discrete topology*. The topology of X containing X and $\varnothing$ only is the *trivial topology*.

Example 3.1.C. The first topology in Example 3.1.B above is the trivial topology on $X = \{a, b, c\}$ and the last topology is the discrete topology. In general, the discrete topology on X is $\mathcal{T} = \mathcal{P}(X)$ (the power set of X).

Example 3.1.D. $X = \mathbb{R}$ and $\mathcal{T} = \mathcal{P}(\mathbb{R})$ form a topological space. Under this topology, by definition, all sets are open. (This does not yield very useful results!)

Definition. Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on X. If $\mathcal{T}' \supset \mathcal{T}$ then we say $\mathcal{T}'$ is *finer* than $\mathcal{T}$; if $\mathcal{T}'$ properly contains $\mathcal{T}$, we say that $\mathcal{T}'$ is *strictly finer* than $\mathcal{T}$ (we also say that $\mathcal{T}$ is *coarser* or *strictly coarser* than $\mathcal{T}'$). The terms larger/smaller and stronger/weaker are also used.

Example 3.1.D. Of the topologies in Example 3.1.B, the coarsest is the first one (upper right; the trivial topology) and the finest is the last one (lower left; the discrete topology). The topology in the upper right is coarser than the topology center right. Not all topologies on a set are "comparable." The topology in the upper right is neither finer nor coarser than the topology in the center, for example.

Definition. In a topological space $(X, \mathcal{T})$, a set U is *closed* if $X \setminus U = U^c \in \mathcal{T}$.

Note 3.1.E. The topology on a set determines which sequences are convergent and which functions are continuous. Some of the tedious results that we have shown so far, such as the fact that the limit of a sequence of real numbers is unique (Theorem 2-1; this result is based on the usual topology on $\mathbb{R}$), may not hold under different topologies. That's part of the reason we needed to justify those claims. We now return to Kirkwood and the usual topology on $\mathbb{R}$.

Note. Next we define open and closed "relative to" a given set. We'll see similarities between the behaviors of open sets and sets which are open relative to a given set.

Definition. Let $A \subset \mathbb{R}$. Then the set V is said to be *open relative to* A if $V = A \cap U$ for some open set U. V is said to be *closed relative to* A if $V = A \cap U$ for some closed set U.

Note 3.1.F. If set $A = \mathbb{R}$, the "open relative to A" is the same as "open," and "closed relative to A" is the same as "closed." We claimed in Note 3.1.B that the only subsets of $\mathbb{R}$ that are both open and closed are $\mathbb{R}$ itself and $\varnothing$. Things are more complicated with relative open and closed. Sets $\varnothing$ and A are both open and closed relative to A (that's not the complicated part). Consider $A = (0,1) \cup [3,4]$. Notice that $(0,1) = A \cap (-1,2) = A \cap [-1,2]$ so that $(0,1)$ is both open and closed relative to A. Also, $[3,4] = A \cap (2,5) = A \cap [2,5]$ so that $[3,4]$ is both open and closed relative to A. We will define what it means for a set of real numbers to be *connected* below. The fact that we can find more sets (that just A and $\varnothing$) that are both open and closed relative to A is related to the fact that $A = (0,1) \cup [3,4]$ is not a connected set of real numbers. In fact, in a metric space setting this idea is used to define a connected space; see my online notes for Complex Analysis 1 (MATH 5510) on Section II.2. Connectedness (notice Definition II.2.1). The next result is analogous to Theorem 3-2, but is for sets open relative to A. Its proof is to be given in Exercise 3.1.16(b).

Theorem 3-4. Let $A \subset \mathbb{R}$. The subsets of A that are open relative to A satisfy:

(a) $\varnothing$ and A are open relative to A.

(b) If $\{V_1, V_2, \ldots, V_n\}$ is a finite collection of sets that are open relative to A, then

$\cap_{i=1}^{n} V_i$ is open relative to A.

(c) If $\{V_\alpha\}$ is any collection of sets that are open relative to A, then $\cup_\alpha V_\alpha$ is open relative to A.

Note. The following theorem is the most important result in this Analysis 1 class!!! It's proof is given in its entirety in the supplement to this section, Supplement. A Classification of Open Sets of Real Numbers.

Theorem 3-5. A set of real numbers is open if and only if it is a countable union of disjoint open intervals.

Note 3.1.G. Theorem 3-5 allows us to completely describe an open set of real numbers in terms of open intervals. If you take a graduate level real analysis class (such as our Real Analysis 1, MATH 5210), then this result will play a central role in the development of the *Lebesgue measure* of a set of real numbers. The idea of "measure of a set" is to generalize the concept of the "length of an interval." For open sets, we can take the measure simply as the sum of the lengths of the countable disjoint open intervals that make up the open set. See my online notes for Real Analysis 1 on Section 2.2. Lebesgue Outer Measure for some introductory material on measure theory. The supplement to those notes Supplement. An Alternate Approach to the Measure of a Set of Real Numbers explicitly uses open intervals and their lengths to set up measure (see Section 3. Outer and Inner Measure in the supplement).

Note. We next define, for a given set of real numbers, some points of special topological interest. Below, we'll use these points to classify sets as open or closed. Many of these same ideas are seen in Complex Variables (MATH 4337/5337); see my online notes for that class on Section 1.11. Regions in the Complex Plane.

Definition. Let $A \subset \mathbb{R}$.

(a) If there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subset A$ then x is called an *interior point* of A. The set of all interior points of A is called the *interior of* A, denoted $\text{int}(A)$.

(b) If for every $\delta > 0$, $(x-\delta, x+\delta)$ contains a point of A and a point not in A, then x is called a *boundary point of* A (x may or may not be in A). The set of all boundary points of A is the *boundary of* A, denoted $b(A)$, or more commonly $\partial(A)$.

(c) If for all $\delta > 0$, $(x - \delta, x + \delta)$ contains a point of A distinct from x, then x is a *limit point* of A.

(d) A point $x \in A$ is called an *isolated point* of A if there exists $\delta > 0$ such that $(x - \delta, x + \delta) \cap A = \{x\}$.

Example 3.1.F. Let $A = (0, 1] \cup \{2\}$. Then $\text{int}(A) = (0, 1)$ and $b(A) = \partial(A) = \{0, 1, 2\}$. The limit points of A are $[0, 1]$. The only isolated point of A is 2.

Note 3.1.H. If x is *not* in set A and is *not* a boundary point of set A then there is some $\delta > 0$ such that $(x - \delta, x + \delta)$ contains no points of A, as we see by negating the definition of "boundary point." Similarly, if $x \in A$ is *not* a boundary point of set A, then there is some $\delta > 0$ such that $(x - \delta, x + \delta)$ contains no points not in set A; that is, $(x - \delta, x + \delta) \subset A$. In this case, x is an interior point of A.

Note. Theorem 3-6 and Corollary 3-6 allow us to classify open and closed sets in terms of the types of points defined above.

Theorem 3-6. A set is closed if and only if it contains all of its boundary points.

Note. The proof of Corollary 3-6 is to be given in Exercise 3.1.15 parts (d) and (e). These proofs will need the following properties of limit and boundary points:
Exercise 3.1.15. **(a)** If $x \notin A$ and x is a limit point of A, then x is a boundary point of A.
(b) If $x \notin A$ and x is a boundary point of A, then x is a limit point of A.
(c) If x is a boundary point of A, then x is a boundary point of A^c.

Corollary 3-6.

(a) A set is closed if and only if it contains all of its limit points.

(b) A set is open if and only if it contains none of its boundary points.

Definition. Let $A \subset \mathbb{R}$. The *closure* of A, denoted $\overline{A}$, is the set consisting of A and its limit points.

Note 3.1.I. A limit point of A may or may not be an element of A. If it is not an element of A, then by Exercise 3.1.15(a), x is a boundary point of A. Notice, then, that $\overline{A}$ is (also) the set consisting of A and its boundary points.

Theorem 3-7. For $A \subset \mathbb{R}$, $\overline{A}$ is closed.

Definition. Let $A \subset \mathbb{R}$. The collection of sets $\{I_\alpha\}$ is said to be a *cover* of A if $A \subset \cup_\alpha I_\alpha$. If each I_α is open, then the collection is called an *open cover* of A. If the cardinality of $\{I_\alpha\}$ is finite, then the collection is a *finite cover*.

Example. The set $\{I_k\} = \{(1/k, 1 - 1/k) \mid k \in \mathbb{N}\}$ is an infinite open cover of $(0, 1)$. Notice that there is no finite subcover of this open cover of $(0, 1)$, since in a finite subset of $\{I_k\}$ there would be an I_m where m is the maximum index of the sets in the finite subset and then the union would exclude $(0, 1/m]$.

Example. The set $\{I_\alpha\} = \{(a/10, b/10) \mid a, b \in \mathbb{Z}, a < b\}$ is an infinite open cover of $[0, 100]$. Notice that there is a finite subcover of this open cover of $[0, 100]$.

Note. The motivation for the definition of a "compact set," given next, is unclear. However, we'll see that compact sets play a fundamental role in analysis. For example, the Extreme Value Theorem from Calculus 1 (which states that a continuous function on an interval of the form $[a, b]$ has a maximum and a minimum on that interval) will be extended from sets of the form $[a, b]$ to compact sets in this class (see Corollary 4-7(b) in Section 4-1. Limits and Continuity). We'll see that $[a, b]$ is an example of a compact set.

Definition. A set A is *compact* if every open cover of A has a finite subcover.

Note 3.1.J. The importance of compact sets lies in the fact that such a set (as I like to put it) "allows us to make a transition from the infinite to the finite." For example, if we have an arbitrary set of real numbers, that set may not have a maximum element; if we have a finite set of real numbers then it definitely has a maximum element. We will see that a compact set mimics this type of behavior of finite sets. Other examples (such as the Extreme Value Theorem mentioned above) will arise throughout this class. Our goal now is to classify compact sets of real numbers in a more tangible way.

Theorem 3-8. Let $\{A_1, A_2, \ldots\}$ be a countable collection of nonempty closed bounded sets of real numbers such that $A_i \supset A_j$ for $i \leq j$. Then $\cap A_i \neq \varnothing$.

Corollary 3-8. Let $\{A_1, A_2, \ldots\}$ be a countable collection of closed bounded sets of real numbers such that $A_i \supset A_j$ if $i < j$. If $\cap_{i=1}^{\infty} A_i = \varnothing$ then $\cap_{i=1}^{N} A_i = \varnothing$ for some $N \in \mathbb{N}$.

Note. We now state the Lindelöf Property, which addresses any open cover of any set of real numbers. It claims that a countable subcover always exits and the proof, which is to be given in Exercise 3.1.17, uses rational numbers. The rational numbers are useful in such a setting because they form a countable set and the rational numbers are so "densely" distributed throughout the real numbers.

Theorem 3-9. The Lindelöf Property.
Let $A \subset \mathbb{R}$. If $\{I_\alpha\}$ is an open cover of A, then some countable subcollection of $\{I_\alpha\}$ covers A.

Note. We now have the equipment to clearly classify a compact set of real numbers. We should mention up-front that the Heine-Borel Theorem holds in $\mathbb{R}$, but there is other settings where it does not hold. We'll discuss this more below in Note 3.1.K.

Theorem 3-10. Heine-Borel Theorem.
If A is a closed and bounded set of real numbers, then A is compact.

Note 3.1.K. In your mathematical travels, you may encounter compact sets in several settings. In many of these settings, closed and bounded sets will be compact. But this may not hold everywhere! For example, in a normed linear space (that is, a vector space with a norm) the closed and bounded set consisting of the closed unit ball (that is, the set of all vectors of norm at most 1 and in standard position) is compact if and only if the space is finite dimensional. Details on this (and a proof) are given in my online notes for Fundamentals of Functional Analysis (MATH 5740) on Section 2.8. Finite Dimensional Normed Linear Spaces; notice Reisz's Theorem (Theorem 2.34). Put more simply, the Heine-Borel Theorem (Theorem 3-10) holds in finite dimensions; that is, in $\mathbb{R}^n$ a closed and bounded set is compact. However, in an infinite dimensional space, the Heine-Borel Theorem does not hold and there are closed and bounded sets that are not compact! An example of such a situation occurs in the infinite dimensional space ℓ^2, the elements of which are square summable sequences. In this space, the set $\{(1, 0, 0, \ldots, (0, 1, 0, \ldots), (0, 0, 1, 0, \ldots), \ldots\}$ is a bounded set (since all the vectors are unit vectors, this set is contained in the unit ball and so is bounded) but there is an open cover of this set which has no finite subcover. That is, the set is closed and bounded but not compact. Details are given in my supplemental note for Fundamentals of Functional Analysis on Supplement. Projections and Hilbert Space Isomorphisms, where a specific open cover illustrating the claim is given and a picture of such a set is presented (with some obvious shortcomings, since the set "lives" in infinite dimensions, but we are stuck here in three dimensions).

Theorem 3-11. A set that is compact is closed and bounded.

Note 3.1.L. After the comments of Note 3.1.K, hopefully you are cautious about extending the results of the class (which concern sets of real numbers) to more general settings. However, Theorem 3-11 carries over to a more general setting, say metric spaces. In a "metric space," we have a collection of points and a way to measure distance. This is sufficient to define a topology on the set of points (called the "metric topology"), and this can be used to discuss open sets, closed sets, and compact sets. In a metric space, a compact set is closed (that is, half of Theorem 3-11 holds; see my online notes for Complex Analysis 1 [MATH 5510] on Section II.4. Compactness; see Proposition II.4.3(a)). In a metric space, a compact set is also "totally bounded" (a property that implies that the set is bounded); this is addressed in the same Complex Analysis 1 notes in Proposition II.4.9.

Note. The next two results also classify compact sets of real numbers. Notice that some of the claims involve limit points of the set. The equivalence claims of Theorem 3-13 parts (a) and (b) follow from the Heine-Borel Theorem (Theorem 3-10) and Theorem 3-11. The equivalence claims of Theorem 3-13 parts (c) and (d) is to be given in Exercise 3.1.19.

Theorem 3-12. A set $A \subset \mathbb{R}$ is compact if and only if every infinite set of points of A has a limit point in A.

Theorem 3-13. Let $A \subset \mathbb{R}$. The following are equivalent.

(a) A is compact.

(b) A is closed and bounded.

(c) Every infinite set of points in A has a limit point in A.

(d) If $\{x_n\}$ is a sequence in A, there is a subsequence $\{x_{n_k}\}$ that converges to some point in A.

Note. The final topic of this section is connectivity. We will see below that a set of real numbers is connected if and only if it is either an interval or a single point (see Theorem 3-1-A).

Definition. A set A is *connected* if there do not exist two open sets U and V such that:

(i) $U \cap V = \varnothing$.

(ii) $U \cap A \neq \varnothing$ and $V \cap A \neq \varnothing$.

(iii) $(U \cap A) \cup (V \cap A) = A$.

Note 3.1.M. We can define the open sets U and V as being a *separation* of set A. A connected set is then a set for which there has no separation. By the way, since this definition of "connected" only refers to open sets and set theoretic manipulations, this definition can be taken verbatim in a topological space. This is done in Introduction to Topology (MATH 4357/5357); see my online notes for that class on Section 23. Connected Spaces. The same approach using a separation can also be taken in the metric space setting; see my online notes for Complex Analysis 1 (MATH 5510) on Section II.2. Connectedness and notice Note II.2.A.

Note 3.1.N. Suppose set A is not connected. Then there is a separation, say U and V, of A. Define sets $A_1 = U \cap A$ and $A_2 = V \cap A$. By part (ii) of the definition of connected, sets A_1 and A_2 are disjoint, nonempty, and $A = A_1 \cup A_2$. By the definition of "open relative to set A," both A_1 and A_2 are open relative to A. Also, since $A_1 = V^c \cap A$ and $A_2 = U^c \cap A$ (where U^c and V^c are closed) then A_1 and A_2 are also closed relative to A. Now $\varnothing$ and A are both open and closed relative to A (see Theorem 3-4(a)), but A_1 and A_2 are also both open and closed relative to A. We claimed in Note 3.1.B that the only subsets of $\mathbb{R}$ that are both open and closed are $\varnothing$ and $\mathbb{R}$ itself. The reason that we can find more sets that are both open and closed relation to set A in this example is related to the fact that A is not connected. In fact, is some settings connectivity is defined in terms of the sets which are both open and closed. This is the case in metric spaces, for example. See my Complex Analysis 1 (MATH 5510) notes on Section II.2. Connectedness and notice Definition II.2.1. See also my notes for Real Analysis 2 (MATH 5220) on Section 11.6. Connected Topological Spaces.

Theorem 3-14. A set of real numbers with more than one element is connected if and only if it is an interval.

Note. A single point forms a connected set, but a single point (which makes up a "singleton") is not an interval (by definition, an interval has distinct endpoints; see Section 1.2. Properties of the Real Numbers as an Ordered Field). This observation, along with Theorem 4-13, gives the following classification of connected sets of real numbers.

Theorem 3-1-A. A set of real numbers is connected if and only if it is a singleton or an interval.

Note 3.1.O. Eduard Heine (March 16, 1821–October 21, 1881) was born in Berlin, attended the University of Berlin for one semester in 1838, and the transfered to the University of Göttingen where he attended lectures by Carl Friedrich Gauss. After three semesters, he returned to the University of Berlin where he took classes from Lejeune Dirichlet, and interacted with other prominent faculty such as Karl Weierstrass and Leopold Kronecker. He finished his thesis work on differential equations in 1842. He then spent a year at Königsberg and in 1844 took at job at the University of Bonn. In 1856 he was promoted and took a job in Halle, Germany at the Martin Luther University Halle-Wittenberg (known as the University of Halle, until it merged with the University of Wittenberg in 1817). He published papers of the summation of series, continued fractions, and elliptic functions. He is best known for the Heine-Borel Theorem (Theorem 3-10), but he also introduced the concept of uniform continuity (which we will study in Section 4.1. Limits and Continuity).

Images from the MacTutor biography webpages of Eduard Heine (left) and Émile Borel (right); this note is base on this webpages

Frenchman Émile Borel (January 7, 1871–February 3, 1956) entered the École Normale in Paris in 1889. In 1893 he was awarded a doctorate, and his thesis title was "On Some Points of the Theory of Functions" which he wrote under the direction of Gaston Darboux. His thesis covered some theory of measure, divergent series, and included a statement and proof of what would become known as the "Heine-Borel Theorem." He took a job at the University of Lille (in northern France, near the Belgium boarder) in 1893 and spent three years there producing many high quality research papers. In 1897 he returned to Paris and joined the faculty at the École Normal Supérieure. In addition, he taught at the Collège de France and in 1905 was elected president of the French Mathematical Society. From 1909 to 1941 he served as a chair of Theory of Functions (a position created for him) at Sorbonne Univeristy (in Paris). Of particular importance to real analysis, Borel introduced the first effective theory of the measure of a set of real numbers. This work, supplemented by work of his fellow Frenchman René Baire and Henri Lebesgue, lead to modern theory measure theory and integration theory. These topics are covered in our graduate level Real Analysis 1 (MATH 5210); see my online notes for Real Analysis 1 for more details. Borel also authored several influential books on such topics as set theory, paradoxes of infinity, mathematical physics, and probability theory. In fact, at the advanced level, probability theory is a branch of measure theory; see my online notes for Measure Theory Based Probability for more information.

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