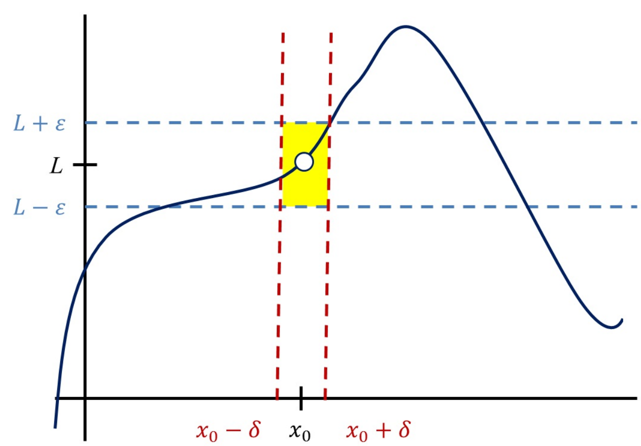
Chapter 4. Continuous Functions.

4.1. Limits and Continuity.

Note. In this section we define limits, continuity, and uniform continuity. We prove several fundamental results of continuous functions, including the Intermediate Value Theorem (Corollary 4-7(b)) and the Extreme Value Theorem (Corollary 4-9). We start with the definition of limit.

Definition. Suppose f is a function of real numbers with domain $\mathcal{D}(f)$. The *limit* of f as x approaches x_0 is L, denoted $\lim_{x\to x_0} f(x) = L$, means:

$$\text{For all } \varepsilon > 0, \text{ there exists } \delta(\varepsilon) > 0 \text{ such that}$$
$$\text{if } 0 < |x - x_0| < \delta(\varepsilon) \text{ and } x \in \mathcal{D}(f), \text{ then } |f(x) - L| < \varepsilon.$$



Note 4.1.A. For my take on an informal understanding of the Calculus 1 definition of limit, see my online notes for that class on Section 2.2. Limit of a Function and Limit Laws; notice "Informal Definition of Limit" and "Dr. Bob's Anthropomorphic Definition of Limit." As Kirkwood states (on his page 73): "The ideas

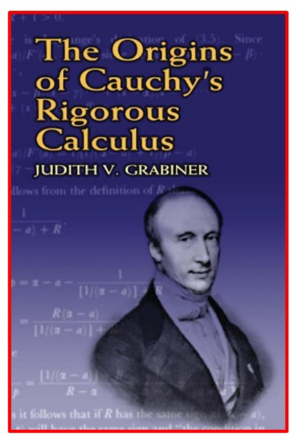
is that $f(x)$ may be made arbitrarily close to L by making x sufficiently close to x_0." In more detail, the "arbitrarily close" idea is given by the fact that ε can be *any* positive value; in particular, it can be really small. The "sufficiently close" idea is given by the fact that $\delta > 0$ is chosen (so to speak) after ε is given. As you might expect, and as you see in Calculus 1 (MATH 1910) when considering "target values" (see Example 2.3.1 in my online supplement for Calculus 1 on Section 2.3. The Precise Definition of Limit—Examples and Proofs) that the value of δ depends on the value of ε, as we indicate above with the notation "$\delta(\varepsilon)$." Notice that, in the definition of $\lim_{x\to x_0} f(x) = L$, we do not require that f is defined at x_0 and that we only are concerned with x values in the domain of f which satisfy $0 < |x - x_0| < \delta$; in particular, we avoid the value $x = x_0$. When considering limits, it "doesn't matter what happens *at* $x = x_0$, but instead what matters is the valued of $f(x)$ for x near (but not equal to) x_0. The ε and δ quantities are what defines this nearness and "close to."

Note 4.1.B. Notice that this definition differs slightly from the one encountered in Calculus 1 (MATH 1910) in that here we require the x values to be in the domain of the function; see my online notes for Calculus 1 on Section 2.3. The Precise Definition of a Limit. One implication of this is that $\displaystyle\lim_{x\to 0}\sqrt{x}$ does not exist in Calculus 1 (since $\sqrt{x}$ is not defined on a deleted open interval centered at 0, as required by the Calculus 1 definition), but with our definition $\displaystyle\lim_{x\to 0}\sqrt{x} = 0$, since we only consider x values in the domain of $\sqrt{x}$. Of course this is just an artifact of the definitions, not that one is right and the other wrong.

Note. You may hear limits described in terms of function values getting "closer and closer" to a limit value (you may even hear that the function "never gets there"). This is a common way try to convey the complicated idea of a limit, but it is not correct! It is not about getting closer and closer; if anything, it is about "getting close and staying close" (namely, $f(x)$ gets within $\varepsilon > 0$ of L and stays there for all $0 < |x - x_0| < \delta$). Since we require $|f(x) - L| < \varepsilon$, there is no prohibition of $f(x)$ taking on the value L (so f can "get there"); there *is* a prohibition against x taking on the value x_0 in the requirement that $0 < |x - x_0|$, which implies $x \neq x_0$.

Note 4.1.C. Our definition is due to the French mathematician Augustin Louis Cauchy (August 21, 1789–May 23, 1857). It is Cauchy that makes the informal ideas of "arbitrarily close" and "sufficiently close" formal in the early 1800s. Calculus grew from a somewhat informal endeavor to a mathematically rigorous area of study in the 1800s, and largely due to Augustin Cauchy. His contributions are spelled out in Judith V. Grabiner's *The Origins of Cauchy's Rigorous Calculus*, MIT Press, 1981 (today, this book is in print by Dover Publications).

Augustin Cauchy



The image of Cauchy is from the MacTutor biography of Cauchy (accessed 2/16/2024).

Before Cauchy, the approach to limits and calculus were less rigorous. Additional history of calculus can be found in my Calculus 1 notes on Section 2.3. The Precise Definition of a Limit. Next, we illustrate the logic of our definition with a specific example.

Example. Prove that $\lim_{x\to a}(mx+b) = ma+b$ where $m \neq 0$.

Note. In Calculus 1, you often compute limits by "factoring, canceling, and substituting"; see my online Calculus 1 notes on Section 2.2 Limit of a Function and Limit Laws (notice Theorem 2.2.A, Dr. Bob's Limit Theorem, and the Note preceding it). This process can be made rigorous, as the following example suggests.

Example 4.1. Let $f(x) = (2x^2-8)/(x-2)$. Prove that $\lim_{x\to 2} f(x) = 8$.

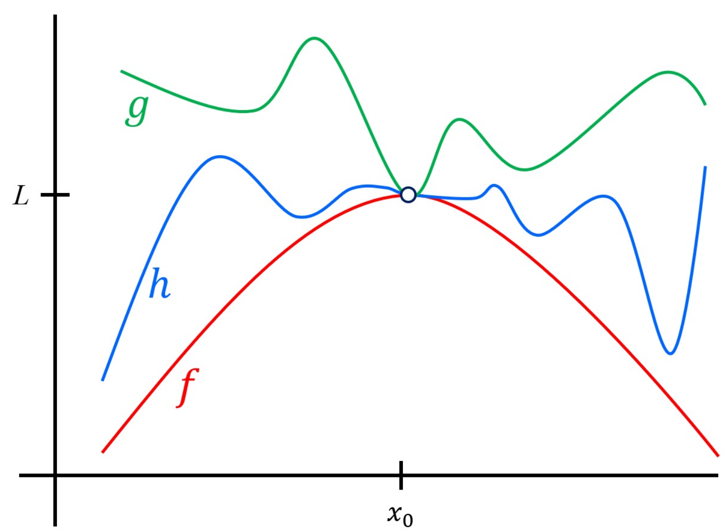
Note. We will often explore limits of functions in terms of the behavior of sequences. The next result relates the limit of a function to limits of certain sequences.

Theorem 4-1. Let f be a function of a real number with domain $\mathcal{D}(f)$ and let x_0 be a limit point of $\mathcal{D}(f)$. Then $\lim_{x\to x_0} f(x) = L$ if and only if for every sequence $\{x_n\} \subset \mathcal{D}(f)$ with $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x_0$, the sequence $\{f(x_n)\}$ converges to L.

Example 4.4. Let $f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$ Prove that $\lim_{x\to 0} f(x)$ does not exist.

Note. The next result if the "Sandwich Theorem" (or "Squeeze Theorem") that you see in Calculus 1 (MATH 1910); for a statement of it see my online Calculus 1 notes on Section 2.2. Limit of a Function and Limit Laws (notice Theorem 2.4), and for a proof see Appendix A.4. Proofs of Limit Theorems. A proof (based on our definition of limit) is to be given in Exercise 4.1.5.

Theorem 4-2. Let f and g be functions with $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$. Suppose h is a function for which $f(x) \leq h(x) \leq g(x)$ if $0 < |x - a| < \delta$ for some $\delta > 0$. Then $\lim_{x\to a} h(x)$ exists and $\lim_{x\to a} h(x) = L$.



Note. In Calculus 1 (MATH 1910), continuity is defined in terms of a limits. There, it is defined differently for interior points and endpoints of the domain; see my online notes on Section 2.5. Continuity. In here, we define continuity in terms of ε's and δ's. Since we only consider values in the domain of f, we need not distinguish between interior points and endpoints.

Definition. Suppose f is a function and $x_0 \in \mathcal{D}(f)$. Then f is *continuous* at x_0 if

$$\text{for all } \varepsilon > 0 \text{ there exists } \delta(\varepsilon) > 0 \text{ such that}$$
$$\text{for all } |x - x_0| < \delta(\varepsilon) \text{ and } x \in \mathcal{D}(f) \text{ we have } |f(x) - f(x_0)| < \varepsilon.$$

If f is not continuous at $x_0 \in \mathcal{D}(f)$, the f is *discontinuous* at x_0.

Note 4.1.D. Of course x_0 must be in $\mathcal{D}(f)$ for f to be continuous at x_0. However, we have not required f to be defined "near" x_0; that is, we have not required x_0 to be a limit point of $\mathcal{D}(f)$. So for any "isolated point" x_0 of $\mathcal{D}(f)$ (that is for any $x_0 \in \mathcal{D}(f)$ such that for some $\delta > 0$ we have $(x_0 - \delta, x_0 + \delta) \cap \mathcal{D}(f) = \{x_0\}$), we have that f is continuous at x_0. This is because the definition of continuity is *vacuously* satisfied at x_0. If x_0 is a limit point of $\mathcal{D}(f)$, then our the definition of continuous implies that $\lim_{x \to x_0} f(x) = f(x_0)$. This is precisely the definition given in Calculus 1 for interior points and endpoints of the domain of f.

Note 4.1.E. For $x_0 \in \mathcal{D}(f)$, by negating the definition of continuous at x_0 we have:

> **Definition.** Suppose f is a function and $x_0 \in \mathcal{D}(f)$. Then f is *not continuous* at x_0 if: there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists $x_* \in \mathcal{D}(f)$ such that $|x_* - x_0| < \delta$ and $|f(x_*) - f(x_0)| \geq \varepsilon$.

Note. Just as we related limits of sequences and limits of functions in Theorem 4-1, we can relate limits of sequences and continuity, as follows.

Corollary 4-2. Let f be a function. Suppose $x_0 \in \mathcal{D}(f)$. Then f is continuous at x_0 if and only if $\lim_{n\to\infty} f(x_n) = f(x_0)$ for all sequences $\{x_n\} \subset \mathcal{D}(f)$ with $\lim_{x\to\infty} x_n = x_0$.

Note 4.1.F. Corollary 4-2 gives that f is continuous at x_0 if and only if, for any sequence $\{x_n\}$ of elements of $\mathcal{D}(f)$ where $\lim_{n\to\infty} x_n = x_0$,

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{x\to\infty} x_n\right) = f(x_0).$$

This is an important observation! In many settings, continuity is dealt with in terms of "passing limits through."

Example 4.7. Let

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \text{ where } \frac{m}{n} \text{ is in lowest terms.} \end{cases}$$

Then f is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous on $\mathbb{Q}$.

Note. We can combine two continuous functions to produce a new continuous function. In particular, as shown in the next theorem (Theorem 4-3), a linear combination of two continuous functions, the product of two continuous functions, and the quotient of continuous functions are each continuous (except where we would get 0 in the denominator of the quotient). In Theorem 4-4 we show that the composition of continuous functions is continuous.

Theorem 4-3. Let f and g be functions with domains $\mathcal{D}(f)$ and $\mathcal{D}(g)$. Suppose $x_0 \in \mathcal{D}(f) \cap \mathcal{D}(g)$ and that f and g are both continuous at x_0. Then

(a) $\alpha f + \beta g$ is continuous at x_0.

(b) fg is continuous at x_0.

(c) f/g is continuous at x_0 if $g(x_0) \neq 0$.

Theorem 4-4. Let f and g be functions with domains $\mathcal{D}(f)$ and $\mathcal{D}(g)$ such that the range of f is a subset of $\mathcal{D}(g)$. If $x_0 \in \mathcal{D}(f)$ and if f is continuous at x_0 and if g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Note 4.1.G. The function $f(x) = a^x$ is defined for all $a > 1$ and x positive in Exercise 3.1.17. For $x = 0$ we define $a^0 = 1$, and for x negative we define $a^x = 1/(a^{-x})$. For $0 < a < 1$ we define as $a^x = (1/a)^{-x}$. With $a = 1$, define $1^x = 1$ for all x. This gives the function $f(x) = a^x$ defined for all $a > 0$ and $x \in \mathbb{R}$. In Exercise 3.1.18, we have that $f(x) = a^x$ is monotone (strictly monotone increasing for $a > 1$ and strictly monotone decreasing for $0 < a < 1$). This is an example of a class of continuous functions, as we show next.

Exercise 4.1.21. Prove that $f(x) = a^x$ is continuous for any $a > 0$.

Note. We next address functions continuous on a set (instead of at a point), and some resulting properties.

Definition. We say that f is *continuous on set A* if f is continuous at each point of set f.

Note. Again, our definition differs slightly from that encountered in Calculus 1 (MATH 1910). For example, in Calculus 1 a function f is continuous on interval $[a, b]$ if it is continuous at each point of interval (a, b) and $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$. See my online notes for Calculus 1 on Section 2.5. Continuity; notice The Continuity Test. In the next theorem, we give a concise classification of continuous functions on a set.

Theorem 4-5. Let $f : X \to Y$. Then f is continuous on $\mathcal{D}(f)$ if and only if for all open sets $A \subset Y$, we have $f^{-1}(A)$ is open relative to $\mathcal{D}(f)$.

Note 4.1.H. In a topological space, Theorem 4-5 is the inspiration for the definition of a continuous function: For function f mapping topological space $(X, \mathcal{T}_1)$ to topological space $(Y, \mathcal{T}_2)$, we say that f is *continuous* if for each open set $A \in \mathcal{T}_2$ we have $f^{-1}(A) \in \mathcal{T}_1$. For more on this, see my online notes for Introduction to Topology (MATH 4357/5357) on Section 18. Continuous Functions. We now have several ways to classify continuous functions. We summarize them in the next theorem.

Theorem 4-6. Let f be a function with domain $\mathcal{D}(f)$. The following are equivalent.

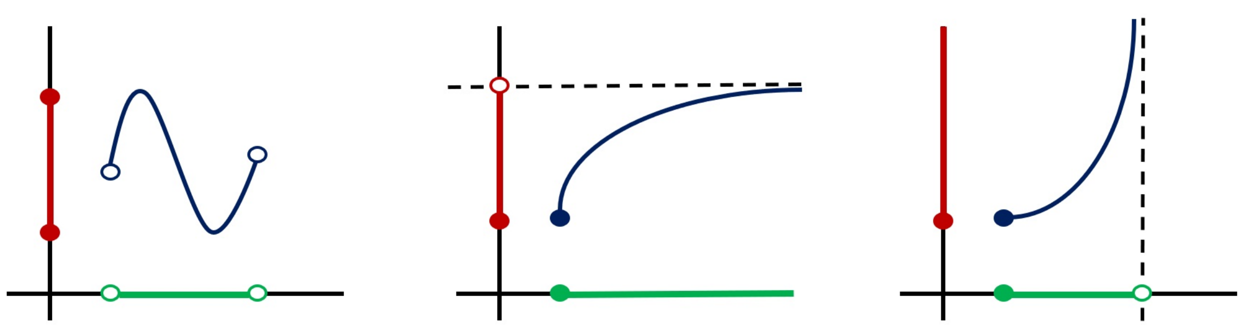
(a) f is continuous on $\mathcal{D}(f)$.

(b) For all $\varepsilon > 0$ and $x_0 \in \mathcal{D}(f)$ there exists $\delta(\varepsilon, x_0) > 0$ such that $|x - x_0| < \delta(\varepsilon, x_0)$ and $x \in \mathcal{D}(f)$ we have $|f(x) - f(x_0)| < \varepsilon$.

(c) If $x_0 \in \mathcal{D}(f)$ and $\{x_n\}$ is a sequence in $\mathcal{D}(f)$ and $\{x_n\} \to x_0$, then $\{f(x_n)\} \to f(x_0)$.

(d) If A is open then $f^{-1}(A)$ is open relative to $\mathcal{D}(f)$.

Note 4.1.I. It might be interesting to explore what properties of a set are preserved by continuous functions. For example, if f is continuous and A is open, then is $f(A)$ open? If f is continuous and B is closed, then is $f(B)$ closed? If f is continuous and C is bounded, then is $f(C)$ bounded? The answer to each of these questions is "no." It turns out that there are two properties of sets which are preserved by **C**ontinuous functions (as we show below): **C**ompactness and **C**onnectedness.



Examples of continuous functions mapping an open set to a closed set, a closed set to a nonclosed set, and a bounded set to an unbounded set.

In the next theorem we address compact sets and continuous functions.

Theorem 4-7. Let $f : X \to Y$ be continuous. If A is a compact subset of X, then $f(A)$ is compact.

Corollary 4-7(a). A continuous function on a compact set of real numbers is bounded.

Note. Since a compact set of real numbers is closed and bounded by the Heine-Borel Theorem (Theorem 3-10) and Theorem 3-11, then Corollary 4-7(a) follows immediately from Theorem 4-7. In Calculus 1 (MATH 1910) when considering "max/min problems," you see the Extreme Value Theorem for Continuous Functions which justifies the existence of the max/min problems you see in Calculus 1. In that setting, the continuous functions are often on closed and bounded intervals, $[a, b]$, and this is the case for the Extreme Value Theorem of Calculus 1. See my online notes for that class on Section 4.1. Extreme Values of Functions on Closed Intervals and notice Theorem 4.1. In this class, we generalize the Calculus 1 version by replacing closed and bounded $[a, b]$ with a general compact set, as follows.

Corollary 4-7(b). The Extreme Value Theorem.
A continuous function on a compact set A attains its maximum and minimum values on A.

Definition. A function that attains its supremum on a set is said to *attain its maximum* on the set. A function that attains its infimum on a set is said to *attain its minimum* on the set. In these cases, the maximum and minimum values are called the *extrema* of the function on the set.

Note 4.1.J. describe continuity

idea of closeness and if a function is positive at a point then it's positive close by

The following... a proof of a more general version is to be given in Exercise 4.1.12.

Theorem 4-8/Corollary 4-8. If f is a continuous function at $x = c$ and if $f(c) > 0$, then there is a $\delta > 0$ such that $f(x) > 0$ if $x \in (c-\delta, c+\delta) \cap \mathcal{D}(f)$. If f is a continuous function at $x = c$ and if $f(c) < 0$, then there is a $\delta > 0$ such that $f(x) < 0$ if $x \in (c-\delta, c+\delta) \cap \mathcal{D}(f)$.

Exercise 4-1-29(a). Prove that if f is a continuous function and A is a connected set, then $f(A)$ is a connected set.

Definition. Let f be a function with domain $\mathcal{D}(f)$. We say that f has the *intermediate value property* if for all $x_1, x_2 \in \mathcal{D}(f)$ with $f(x_1) < f(x_2)$ and $y \in (f(x_1), f(x_2))$, there exists $x_3 \in \mathcal{D}(f)$ with x_3 between x_1 and x_2 and $f(x_3) = y$.

Theorem 4-9. If f is a continuous function of $[a, b]$ and if $f(a)$ and $f(b)$ are of opposite signs, then there is a number $c \in (a, b)$ for which $f(c) = 0$.

Corollary 4-9. Intermediate Value Theorem.
Let f be continuous on an $[a, b]$ and α is a number between $f(a)$ and $f(b)$, then there is a number $c \in (a, b)$ where $f(c) = \alpha$.

Definition. Let f be continuous with domain $\mathcal{D}(f)$ and $A \subset \mathcal{D}(f)$. Then f is *uniformly continuous* on A if

$$\text{for all } \varepsilon > 0, \text{ there exists } \delta(\varepsilon) > 0 \text{ such that}$$
$$\text{if } x, y \in A \text{ and if } |x - y| < \delta \text{ then } |f(x) - f(y)| < \varepsilon.$$

Note. The idea of uniform continuity is that the choice of δ depends on ε only, and not on x.

Theorem 4-10. If f is continuous on a compact set A, then f is uniformly continuous on A.

Corollary 4-10. A continuous function on a compact set A is uniformly continuous on A.