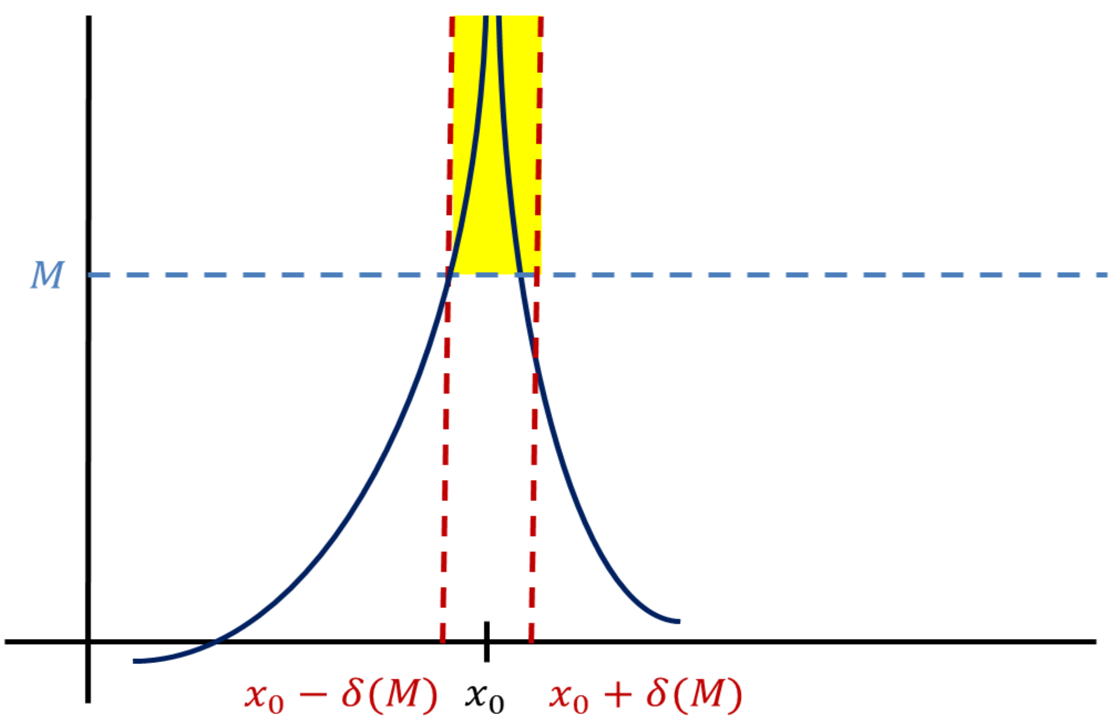
4.2. Monotone and Inverse Functions.

Note. In this section we further explore the idea of a limit and consider infinite limits and one-sided limits. We look at the interaction of functions and sequences. Monotone functions are defined and various types of discontinuities are defined.

Definition. Suppose f is a function with domain $\mathcal{D}(f)$ and let x_0 be a limit point of $\mathcal{D}(f)$. Then the *limit of* f *as* x *approaches* x_0 *is* ∞ means:

$$\text{For all } M \in \mathbb{R}, \text{ there exists } \delta(M) > 0 \text{ such that}$$
$$\text{if } 0 < |x - x_0| < \delta(M) \text{ and } x \in \mathcal{D}(f), \text{ then } f(x) > M.$$

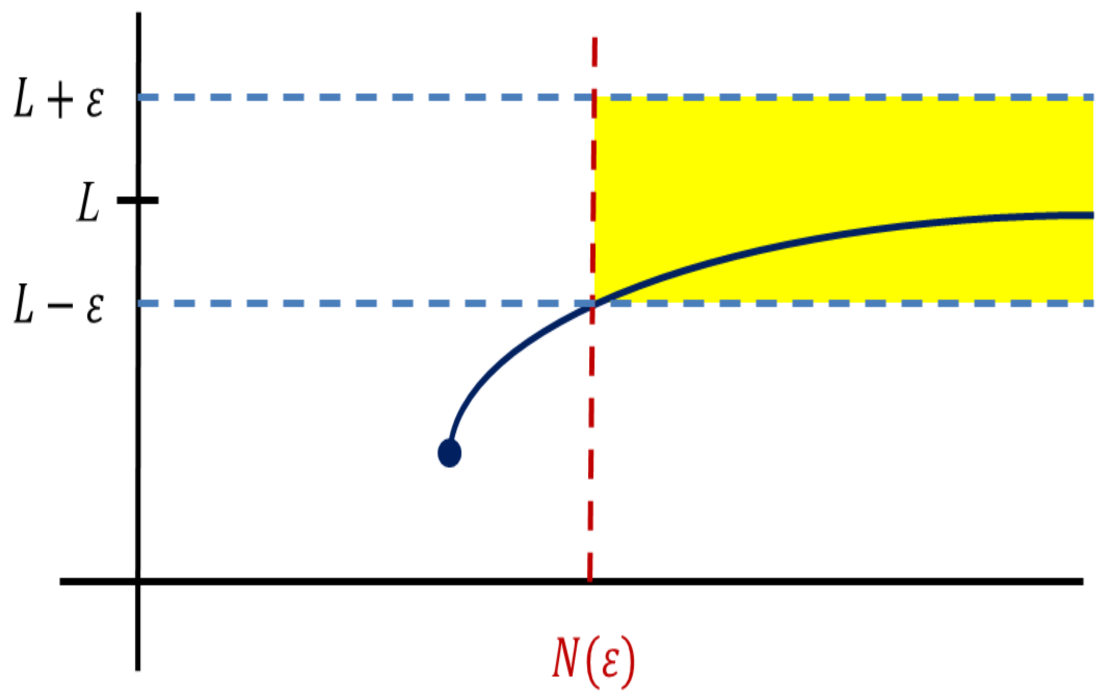
We denote the as $\lim_{x \to x_0} f(x) = \infty$. We can similarly define $\lim_{x \to x_0} f(x) = -\infty$.



Definition. Suppose f is a function with domain $\mathcal{D}(f)$ where $\mathcal{D}(f) \cap (M, \infty) \neq \emptyset$ for all $m \in \mathbb{R}$. Then we say that the *limit of f as x goes to ∞ is L* means:

$$\text{for all } \varepsilon > 0 \text{ there exists } N(\varepsilon) \text{ such that}$$
$$x < N(\varepsilon) \text{ and } x \in \mathcal{D}(f) \text{ implies } |f(x) - L| < \varepsilon.$$

We denote the as $\lim_{x\to\infty} f(x) = L$.



Definition. Let f have domain $\mathcal{D}(f)$ and let x_0 be a limit point of $\mathcal{D}(f) \cap [x_0, \infty)$. Then the *limit of f as x approaches x_0 from the right if L* means:

$$\text{for all } \varepsilon > 0 \text{ there exists } \delta(\varepsilon) > 0 \text{ such that}$$
$$0 < x - x_0 < \delta(\varepsilon) \text{ and } x \in \mathcal{D}(f) \text{ implies } |f(x) - L| < \varepsilon.$$

We denote this as $\lim_{x\downarrow x_0} f(x) = L$ or $\lim_{x\to x_0^+} f(x) = L$.

Note. We can similarly define $\lim_{x \uparrow x_0} f(x) = L$ or $\lim_{x \to x_0^-} f(x) = L$.

Theorem 4-11. Let f be a function with domain $\mathcal{D}(f)$ and suppose x_0 is a limit point of both $\mathcal{D}(f) \cap [x_0, \infty)$ and $\mathcal{D}(f) \cap (-\infty, x_0]$. Then $\lim_{x \to x_0} f(x) = L$ if and only if $\lim_{x \downarrow x_0} f(x) = L$ and $\lim_{x \uparrow x_0} f(x) = L$.

Definition. Let f be a function with domain $\mathcal{D}(f)$ and suppose $x_0 \in \mathcal{D}(f)$. We say f is *continuous from the right* at x_0 if

$$\text{for all } \varepsilon > 0, \text{ there exists } \delta(\varepsilon) > 0 \text{ such that}$$
$$\text{for all } x \text{ with } 0 \le n - x_0 < \delta(\varepsilon) \text{ and } x \in \mathcal{D}(f), \text{ we have } |f(x) - f(x_0)| < \varepsilon.$$

Continuity from the left at x_0 is similarly defined.

Corollary 4-11. Let f have domain $\mathcal{D}(f)$ with $x_0 \in \mathcal{D}(f)$. Then f is continuous at x_0 if and only if it is continuous from the right and left at x_0.

Theorem 4-12. Let f have domain $\mathcal{D}(f)$ with $x_0 \in \mathcal{D}(f)$. Then f is continuous from the right if and only if for all sequences $\{x_n\} \subset \mathcal{D}(f)$ with $x_n \ge x_0$ and $\{x_n\} \to x_0$, we have $\{f(x_n)\} \to f(x_0)$.

Corollary 4-12. Let f have domain $\mathcal{D}(f)$ with $x_0 \in \mathcal{D}(f)$. Then f is continuous from the right at x_0 if an only if for all decreasing sequences $\{x_n\} \subset \mathcal{D}(f)$ with $\{x_n\} \to x_0$, we have $\{f(x_n)\} \to f(x_0)$.

Definition. A function f is *monotone increasing* if for any $x_1, x_2 \in \mathcal{D}(f)$ with $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$. If for any $x_1 < x_2$ in $\mathcal{D}(f)$, we have $f(x_1) < f(x_2)$, then f is *strictly monotone increasing.*

Theorem 4-13. Let f be monotone with domain an open interval (a, b). Then $\lim_{x \uparrow x_0} f(x)$ and $\lim_{x \downarrow x_0} f(x)$ exist for all $x_0 \in (a, b)$.

Theorem 4-14. A monotone function with domain an open interval can have at most countably many discontinuities.

Definition. A function f has a *removable discontinuity* at x_0 if $\lim_{x \to x_0} f(x)$ exists, but $\lim_{x \to x_0} f(x) \neq f(x_0)$ or $f(x_0)$ does not exist.

Definition. A function f has a *discontinuity of the third type* if f is not continuous at x_0, but it does not have a jump or removable discontinuity.

Example. Functions $f(x) = 1/x^2$ and $g(x) = \sin(1/x)$ both have discontinuities of the third kind at $x_0 = 0$.

Theorem 4-15. If f is defined on (a, b) and satisfies the intermediate value property on (a, b), then f cannot have a removable nor a jump discontinuity on (a, b).

Theorem 4-16. Continuity of the Inverse Function.

Suppose f is continuous and strictly monotone on $[a, b]$. Then

(a) f^{-1} is strictly monotone on its domain.

(b) f^{-1} is continuous on its domain.

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