Chapter 5. Differentiation.

5.1. The Derivative of a Function.

Note. In this section we define the derivative of a function and explore some of its properties.

Definition. Let f be a function defined on an interval (a, b) and suppose $c \in (a, b)$. We say f is *differentiable* at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

exists and is finite. If so, we say the limit is the *derivative* of f at c, denoted $f'(c)$.

Note. The following result show that differentiability at a point can be dealt with in terms of limits of sequences.

Theorem 5-1. Let f be defined on (a, b) with $c \in (a, b)$. Then f is differentiable at c if and only if for every sequence $\{x_n\} \subset \mathcal{D}(f)$ with $\{x_n\} \to c$ and $x_n \neq c$ for all n,

$$\lim_{n \to \infty} \frac{f(x_n) - f(c)}{x - c} = f'(c).$$

Note. The following result is seen in Calculus 1 and, simply put, states that "differentiability implies continuity."

Theorem 5-2. Let f be defined on (a, b) and suppose f is differentiable at $c \in (a, b)$. Then f is continuous at c.

Theorem 5-3. Linearity, Product Rule, Quotient Rule.
Let f and g be defined on (a, b) and differentiable at $c \in (a.b)$. Then

(a) $(\alpha f + \beta g)'(c) = \alpha f'(c) + \beta g'(c)$ for all $\alpha, \beta \in \mathbb{R}$.

(b) $(fg)'(c) = f(c)g'(c) + g(c)f'(c)$.

(c) $\left(\dfrac{f}{g}\right)' = \dfrac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$ if $g'(c) \neq 0$.

Note. Theorem 5-3(a) implies that a linear combination of differentiable functions is differentiable or, equivalently, that the differentiation operator is linear. If we put the derivative of a function in square brackets, we can sort of draw a picture of the product and quotient rules as follows:

$$D[(f)(g)] = [f'](g) + (f)[g'] \qquad D\left[\frac{(f)}{(g)}\right] = \frac{[f'](g) - (f)[g']}{(g)^2}$$

where we use D to represent the differentiation operator. These "pictures" can be made even more schematic:

$$D[(\ )(\ )] = [\ ](\ ) + (\ )[\ ] \qquad D\left[\frac{(\ )}{(\ )}\right] = \frac{[\ ](\ ) - (\ )[\ ]}{(\ )^2}.$$

For more details, see my "A Useful Notation for Rules of Differentiation," *The College Journal of Mathematics*, **24**(4) (1993) 351–352 (which I have posted online at `http://faculty.etsu.edu/gardnerr/pubs/T1.pdf`).

Theorem 5-4. Chain Rule.

Let f be defined on (a, b) and suppose $f'(c)$ exists for some $c \in (a, b)$. Suppose g is defined on an open interval containing the range of f and suppose g is differentiable at $f(c)$. Then $g \circ f = g(f)$ is differentiable at c and $(g \circ f)'(c) = g'(f(c))[f'(c)]$.

Theorem 5-5. Let f be a function defined on (a, b). Suppose that f is differentiable at $c \in (a, b)$ and $f'(c) > 0$. Then there is a number $\delta > 0$ such that $f(c) < f(x)$ if $x \in (c, c + \delta)$ and $f(x) < f(c)$ if $x \in (c - \delta, c)$.

Note. Notice that Corollary 5-5 gives the corresponding result for the case $f'(c) < 0$. Notice that Theorem 5-5 does not say that if $f'(c) > 0$ then f is increasing on some interval about $x = c$, as can be shown by considering

$$f(x) = \begin{cases} x + 2x^2 \sin(1/x), & x \neq 0 \\ 0 & x = 0 \end{cases}$$

at $c = 0$. To insure that f is increasing on an interval, we need $f' > 0$ on the interval.

Theorem 5-6. Suppose f is defined on (a, b) and has a relative extremum at $x_0 \in (a, b)$. If f is differentiable at x_0, then $f'(x_0) = 0$.

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