5.2. Some Mean Value Theorems.

Note. In this section we state and prove the Mean Value Theorem and a generalization of it. We state Taylor's Theorem, the proof of which is based on the Mean Value Theorem, and state and prove L'Hôpital's Rule. Much of the material form this section is the same as encountered in Calculus 1 (MATH 1910).

Theorem 5-1. Rolle's Theorem.

Suppose f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there is $c \in (a, b)$ such that $f'(c) = 0$.

Theorem 5-8. Mean Value Theorem.

Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Corollary 5-8.

(a) Suppose f is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = 0$ for all $x \in (a, b)$ then f is constant on $[a, b]$.

(b) If f and g are continuous on $[a, b]$ and differentiable on (a, b) and $f'(x) = g'(x)$ for $x \in (a, b)$, then f and g differ by a constant on $[a, b]$.

Theorem 5-9. Generalized Mean Value Theorem.

If f and g are continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ with

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

Note. Theorem 5-9 reduces to the Mean Value Theorem with $g(x) = x$.

Note. The following result should remind you of the power series representation of a function. However, notice that it in fact deals with a polynomial approximation of a value of function f at a point b "near" point a where the value of $f(a)$ is known.

Theorem 5-10. Taylor's Theorem.

Suppose f is $n + 1$ time differentiable on an open interval containing $[a, b]$. Then there exists $c \in (a, b)$ such that

$$f(b) = f(a) + f'(a)(b - a) + \cdots + \frac{f^{(n)}(a)}{n!}(b - a)^n + \frac{f^{(n+1)}(c)}{(n + 1)!}(b - a)^{n+1}.$$

Corollary 5-10. Taylor's Theorem (Alternative Version).

Suppose f is $n + 1$ times differentiable on an open interval containing $[a, b]$. Then if $x \in (a, b)$, there exists $c \in (a, x)$ such that

$$f(x) = f(a) + f'(a)(x - a) + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1}.$$

Example. We can use Corollary 5-10 to, for example, approximate the value of $f(x) = \cos(x)$ at $x = 61°$, since we know the value precisely of f at $a = 60°$. Of course, computations must be done in terms of radians.

Theorem 5-10. L'Hôpital's rule.

Let f and g be differentiable on (a, b) with $g'(x) \neq 0$ for all $x \in (a, b)$.

(a) If $\lim_{x \downarrow a} f(x) = 0$, $\lim_{x \downarrow a} g(x) = 0$, and $\lim_{x \downarrow a} \dfrac{f'(x)}{g'(x)} = L$, then $\lim_{x \downarrow a} \dfrac{f(x)}{g(x)} = L$.

(b) If $\lim_{x \downarrow a} f(x) = \pm\infty$, $\lim_{x \downarrow a} g(x) = \pm\infty$, and $\lim_{x \downarrow a} \dfrac{f'(x)}{g'(x)} = L$, then $\lim_{x \downarrow a} \dfrac{f(x)}{g(x)} = L$.

Note. There are many other versions of L'Hôpital's Rule. We may have a replaces with $-\infty$ and the results still hold. We may replace "$\lim_{x \downarrow a}$" with "$\lim_{x \uparrow b}$" and the results still hold (even when $b = \infty$). The results hold when L is replaced with $\pm\infty$.

Theorem 5-12. Suppose f is a strictly monotone continuous function on (a, b). If $f'(x_0)$ exists and is not 0 for some $x_0 \in (a, b)$, then f^{-1} is differentiable at $f(x_0)$ and

$$(f^{-1})'(f(x_0) = \frac{1}{f'(x_0)}.$$

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