Chapter 6. Integration.

6.1. The Riemann Integral.

Note. In this section we will consider a function f which is bounded and defined on an interval $[a, b]$. The results of this section are the **single most important results** covered in Analysis 2 (MATH 4227/5227) from the perspective of a graduate analysis class. We will introduce the Riemann integral of f and give necessary and sufficient conditions for f to be Riemann integrable on $[a, b]$ in terms of the "level of continuity" of f. This result is called the Riemann-Lebesgue Theorem (Theorem 6-11). In the process, we will define a *measure zero* set. The topic of measure theory in general is covered in a graduate level Real Analysis 1 class (our MATH 5210). In fact, I use a version of the notes for this section as an introduction to my Real Analysis 1 class. For details, see `http://faculty.etsu.edu/gardnerr/5210/Riemann-Lebesgue-Theorem.pdf` for a version of the notes from this section, including several proofs.

Theorem 6-1. If A and B are bounded sets with $A \subset B$, then $\sup(A) \leq \sup(B)$ and $\inf(A) \geq \inf(B)$.

Theorem 6-2.

(a) Suppose A and B are nonempty sets such taht $x \in A$ and $y \in B$ implies $x \leq y$. Then $\sup(A)$ and $\inf(B)$ are finite and $\sup(A) \leq \inf(B)$.

(b) Suppose that A and B are nonempty sets such that $x \in A$ and $y \in B$ implies $x \leq y$. Then $\sup(A) = \inf(B)$ if and only if for all $\varepsilon > 0$, there exists $x(\varepsilon) \in A$ and $y(\varepsilon) \in B$ such that $y(\varepsilon) - x(\varepsilon) < \varepsilon$.

Definition. A *partition* $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a, b]$ is a set such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

For partitions P and Q of $[a, b]$, we say that Q is a *refinement* of P if $P \subset Q$.

Definition. Let $x_{i-1}, x_i \in P$, where P is a partition of $[a, b]$. For f a bounded function on $[a, b]$, define

$$m_i(f) = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\},$$

$$M_i(f) = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\},$$

and $\Delta x_i = x_i - x_{i-1}$. Let

$$\overline{S}(f; P) = \sum_{i=1}^{n} M_i(f)\Delta x_i \text{ and } \underline{S}(f; P) = \sum_{i=1}^{n} m_i(f)\Delta x_i.$$

$\overline{S}(f; P)$ and $\underline{S}(f; P)$ are the *upper Riemann sum* and *lower Riemann sum*, respectively, of f on $[a, b]$ with respect to partition P.

Definition. With the notation above, suppose $\overline{x}_i \in [x_{i-1}, x_i]$. Then

$$S(f;P) = \sum_{i=1}^{m} f(\overline{x}_i)\Delta x_i$$

is a *Riemann sum* of f on $[a,b]$ with respect to partition P.

Theorem 6-3.

(a) Suppose P and Q are partitions of $[a,b]$ and Q is a refinement of P. Then $\underline{S}(f;P) \le \underline{S}(f;Q)$ and $\overline{S}(f;P) \ge \overline{S}(f;Q)$.

(b) If P and Q are any partitions of $[a,b]$ then $\underline{S}(f;P) \le \overline{S}(f;Q)$.

(c) Let $\underline{S}(f) = \sup\{\underline{S}(f;P) \mid P$ is a partition of $[a,b]\}$ and $\overline{S}(f) = \inf\{\overline{S}(f;P \mid P$ is a partition of $[1,b]\}$. Then $\underline{S}(f)$ and $\overline{S}(f)$ are finite and $\underline{S}(f) \le \overline{S}(f)$.

Definition. Let f be bounded on $[a,b]$. Then f is said to be *Riemann integrable* on $[a,b]$ if $\overline{S}(f) = \underline{S}(f)$. In this case, $\overline{S}(f)$ is called the *Riemann integral of f on* $[a,b]$, denoted

$$\overline{S}(f) = \int_a^b f(x)\,dx = \int_a^b f.$$

Note. First, let's explore some conditions related to the integrability of f on $[a,b]$. Notice that these conditions are merely restatements of the definition and that the proofs follow from this definition, along with properties of suprema and infima.

Theorem 6-4. Riemann Condition for Integrability.

A bounded function f defined on $[a, b]$ is Riemann integrable on $[a, b]$ if and only if for all $\varepsilon > 0$, there exists a partition $P(\varepsilon)$ of $[a, b]$ such that

$$\overline{S}(f; P(\varepsilon)) - \underline{S}(f; P(\varepsilon)) < \varepsilon.$$

Theorem 6-5. Suppose f is Riemann integrable on $[a, b]$. If I is a number such that $\underline{S}(f; P) \leq I \leq \overline{S}(f : P)$ for every partition P, then $\displaystyle\int_a^b f = I$.

Definition. Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$. The *norm* (or *mesh*) of P, denoted $\|P\|$, is

$$\|P\| = \max\{x_i - x_{i-1} \mid i = 1, 2, 3, \ldots, n\}.$$

Theorem 6-6. A bounded function f is Riemann integrable on $[a, b]$ if and only if for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if P is a partition with $\|P\| < \delta(\varepsilon)$ then

$$\overline{S}(f; P) - \underline{S}(f; P) < \varepsilon.$$

Note. The following result is proved in Calculus 1. In fact, all functions encountered in the setting of integration in Calculus 1 involve continuous functions. We give a proof based on other stated results.

Theorem 6-7. If f is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$.

Note. We now introduce a new idea about the "weight" of a set. We will ultimately see that the previous result gives us, in some new sense, a classification of Riemann integrable functions.

Definition. The (Lebesgue) *measure* of an open interval (a, b) is $b - a$. The measure of an unbounded open interval is infinite. The measure of an open interval I is denoted $m(I)$.

Definition. A set $E \subset \mathbb{R}$ has *measure zero* if for all $\varepsilon > 0$, there is a countable collection of open intervals $\{I_1, I_2, I_3, \ldots\}$ such that

$$E \subset \bigcup_{i=1}^{\infty} I_i \text{ and } \sum_{i=1}^{\infty} m(I_i) < \varepsilon.$$

Note. Recall that if $|r| < 1$ then $\displaystyle\sum_{i=1}^{\infty} r^i = \frac{r}{1-r}$.

Note. The following two results follow from the definition of measure zero.

Theorem 6-8. A subset of a set of measure zero has measure zero.

Theorem 6-9. The union of a countable collection of sets of measure zero is a set of measure zero.

Corollary 6-9. A countable set has measure 0.

Note. It is not the case that cardinality and measure are closely related. The converse of Corollary 6-9, for example, is not true. That is, there exists an uncountable set which is also of measure zero. Such a set is the Cantor (ternary) set.

Definition. Let

$$\begin{aligned}
I &= [0,1] \\
C_1 &= \left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right] \\
C_2 &= \left[0,\frac{1}{9}\right] \cup \left[\frac{2}{9},\frac{1}{3}\right] \cup \left[\frac{2}{3},\frac{7}{9}\right] \cup \left[\frac{8}{9},1\right] \\
C_3 &= \left[0,\frac{1}{27}\right] \cup \left[\frac{2}{27},\frac{1}{9}\right] \cup \left[\frac{2}{9},\frac{7}{27}\right] \cup \left[\frac{8}{27},\frac{1}{3}\right] \cup \left[\frac{2}{3},\frac{19}{27}\right] \cup \left[\frac{20}{27},\frac{7}{9}\right] \cup \left[\frac{8}{9},\frac{25}{27}\right] \cup \left[\frac{28}{27},1\right] \\
&\vdots \\
C_n &= 2^n \text{ intervals, each of length } \frac{1}{3^n} \\
&\vdots
\end{aligned}$$

Define the *Cantor set* to be

$$C = \bigcap_{n=1}^{\infty} C_n.$$

Note. Notice that

$$\begin{aligned}
C_1^c &= \left(\frac{1}{3}, \frac{2}{3}\right) \\
C_2^c &= \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \\
C_3^c &= \left(\frac{1}{27}, \frac{2}{27}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{27}, \frac{8}{27}\right) \cup \left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{19}{27}, \frac{20}{27}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \cup \left(\frac{25}{27}, \frac{28}{27}\right) \\
&\vdots \\
C_n^c &= 2^{n-1} \text{ disjoint intervals, the sume of the lengths of which is } 1 - \frac{1}{3^n} \\
&\vdots
\end{aligned}$$

and so

$$\begin{aligned}
m(C_1^c) &= \frac{1}{3} \\
m(C_2^c) &= \frac{1}{3} + \frac{2}{9} = \frac{5}{9} = 1 - \frac{4}{9} \\
m(C_3^c) &= \frac{1}{3} + \frac{2}{9} + \frac{4}{27} = \frac{19}{27} = 1 - \frac{8}{27} \\
&\vdots \\
m(C_n^c) &= 1 - \frac{2^n}{3^n} \\
&\vdots
\end{aligned}$$

We have

$$\left(\bigcap_{n=1}^{\infty} C_n\right) \bigcup \left(\bigcup_{n=1}^{\infty} C_n^c\right) = [0, 1].$$

So $C^c = \cup_{n=1}^{\infty} C_n^c$. Notice that $C_n^c \subset C_{n+1}^c$. In fact, $\lim_{n\to\infty} m(C_n^c) = 1$. Therefore (well, with some details omitted in this class), $m(C) = 0$.

Note. The Cantor set includes all real numbers between 0 and 1 with base 3 ("ternary") decimal expansion which includes only 0's and 2's. So, C contains all possible "sequences" of 0's and 2's.

Note/Theorem. The Cantor set C is uncountable.

Proof. For $x \in C$ with ternary "decimal" expansion

$$x = 0.(x_1)(x_2)(x_3)\cdots_3$$

define

$$f(x) = 0.\left(\frac{x_1}{2}\right)\left(\frac{x_2}{2}\right)\left(\frac{x_3}{2}\right)\cdots_2.$$

Then $\{f(x) \mid x \in C\}$ will include all "sequences" of 0's and 1's. That is, this set includes the binary representation of all numbers form $[0,1]$. So f is an onto function from C to $[0,1]$. Therefore, C is uncountable. ∎

Note. f as defines above is not one to one (but it is onto). For example,

$$f\left(\frac{2}{3}\right) = f(0.2000\cdots_3) = 0.1000\cdots_2 = \frac{1}{2}$$

and

$$f\left(\frac{1}{3}\right) = f(0.0222\cdots_3) = 0.0111\cdots_2 = \frac{1}{2}.$$

There's a problem with multiple representations when using decimal expansions (though our cardinality claim still holds). As an additional example, $1 = 0.222\cdots_3$ and $1 = 0.999\cdots_{10}$.

Note. We now return to Riemann integrable functions and, ultimately, classify Riemann integrable bounded functions in term of a measure zero set.

Definition. The *oscillation* of a function f on a set A is

$$\sup\{|f(x) - f(y)| \mid x, y \in A \cap \mathcal{D}(f)\}$$

where $\mathcal{D}(f)$ denotes the domain of f. The *oscillation of f at x* is

$$\lim_{h \to 0^+} \left(\sup\{|f(x') - f(x'')| \mid x', x'' \in (x-h, x+h) \cap \mathcal{D}(f)\} \right),$$

denoted $\operatorname{osc}(f; x)$.

Example. Consider

$$f(x) = \begin{cases} x^2 + 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ x^2 - 1 & \text{for } x < 0. \end{cases}$$

has $\operatorname{osc}(f; 0) = 2$. For $x \neq 0$, $\operatorname{osc}(f; x) = 0$.

Theorem 6-10. A function f is continuous at $x \in \mathcal{D}(f)$ if and only if $\operatorname{osc}(f; x) = 0$.

Note. Now for our main result concerning Riemann integrable functions.

Theorem 6-11. The Riemann Lebesgue Theorem

Consider a bounded function f defined on $[a, b]$. If f is Riemann integrable on $[a, b]$ if and only if the set of discontinuities of f on $[a, b]$ has measure zero.

Note. We need a preliminary result before proving the complete Riemann-Lebesgue Theorem.

Exercise 6.1.8. *Let f be a function with $\mathcal{D}(f) = [a, b]$. Then for any $s > 0$,*

$$A_s = \{x \in [a, b] \mid \ osc(f; x) \geq s\}$$

is compact.

Note. With the notation from this exercise, if f is discontinuous at some $x_0 \in [a, b]$, then $x_0 \in A_s$ for some $s > 0$. So for $f : [a, b] \to \mathbb{R}$, the set of discontinuities is $D = \bigcup_{n=1}^{\infty} A_{1/n}$. That is, the set of discontinuities of $f : [a, b] \to \mathbb{R}$ is a *countable union of closed sets*. Such a set is said to be F_σ. One can also show that, more generally, if $f : \mathbb{R} \to \mathbb{R}$, then the set of discontinuities is an F_σ set. Now for the remaining part of the proof of the Riemann-Lebesgue Theorem (Theorem 6-11).

Note. Theorems 6-4 and 6-6 also give necessary and sufficient conditions for the Riemann integrability of a bounded function. However there is, in a sense, no new information in these results since they are really just restatements of the definition of Riemann integral. On the other hand, the Riemann-Lebesgue Theorem cleanly classifies Riemann integrable functions. It gives a condition *on the function, in terms of properties of the function* without any reference to partitions or Riemann sums (directly, at least).

Note. We know from Theorem 6-7 a continuous function f on $[a, b]$ is Riemann integrable on $[a, b]$. So, perhaps, it is not surprising that necessary and sufficient conditions for Riemann integrability of f involve the "level of discontinuity" of f. Informally, interpret the Riemann-Lebesgue Theorem as saying that a function is Riemann integrable if and only if the function is not *too badly discontinuous.*

Example. In Section 4-1, we say an example of a function which is continuous on the rational numbers and discontinuous on the irrational numbers:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 1/q & \text{if } x = p/q \in \mathbb{Q} \end{cases}$$

where p/q are in reduced terms. The set of discontinuities of f has measure zero (since $\mathbb{Q}$ is countable and hence, by Corollary 6-9, measure zero). So f is Riemann integrable on any interval and, in fact, $\int_{\mathbb{R}} f = 0$.

Example. The *Dirichlet function* is:

$$D(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q}. \end{cases}$$

D is bounded (on $[0,1]$, say) but discontinuous on $[0,1]$. In graduate level Real Analysis 1 (MATH 5210), you will see that the measure of $[0,1]$ is 1 (we already know that the measure of $(0,1)$ is 1 but, technically, we cannot *prove* anything about the measure of a closed interval, since we have not yet dealt with measure theory). So D is not Riemann integrable on $[0,1]$. Notice that D is 0 except on the rationals and we know that the rationals are a measure zero set. So, if we can define $\int_0^1 D$, then it *should* be 0. We take this as a first motivation to study another type of integration—one which makes use of the measure of sets. This is a prime motivation to explore another type of integration, known as *Lebesgue integration*. This is the main topic of out Real Analysis 1 (MATH 5210) graduate class. You should take this class, eh!

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