6.2. Some Properties and Applications of the Riemann Integral.

Note. In this section we explore several properties of the Riemann integral, many of which you are familiar with from calculus.

Note. We have seen that, with the notation of Section 6.1,

$$\begin{aligned} M_i(f+g) &= M_i(f) + M_i(g) \\ m_i(f+g) &= m_i(f) + m_i(g) \end{aligned}$$

If $c \geq 0$, $M_i(cf) = cM_i(f)$, and $m_i(cf) = cm_i(f)$. If $c < 0$, $M_i(cf) = cm_i(f)$, and $m_i(cf) = cM_i(f)$.

Theorem 6-13. Suppose f and g are Riemann integrable on $[a,b]$. Then:

(a) $f+g$ is Riemann integrable on $[a,b]$ and $\displaystyle\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

(b) For any number c, cf is Riemann integrable on $[a,b]$ and $\displaystyle\int_a^b cf = c\int_a^b f$.

Note. The Riemann-Lebesgue Theorem (Theorem 6-11) tells us that $f+g$ is Riemann integrable. We need only verify the *value* of the integral.

Theorem 6-14. f is Riemann integrable on $[a, b]$ if and only if f is Riemann integrable on $[a, c]$ and $[c, b]$ for all $c \in (a, b)$. Then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Definition. If f is Riemann integrable on $[a, b]$, then define $\displaystyle\int_b^a f = -\int_a^b f$.

Corollary 6-14. If a, b, c are such that $\int_a^b f$, $\int_b^c f$, and $\int_a^c f$ exist, then

$$\int_a^c f + \int_c^b f = \int_c^b f.$$

Theorem 6-15. Suppose f is Riemann integrable on $[a, b]$ and $f \geq 0$. Then $\displaystyle\int_1^b f \geq 0$.

Corollary 6-15a. If f and g are Riemann integrable on $[a, b]$ and $f \geq g$ then $\displaystyle\int_a^b f \geq \int_a^b g$.

Corollary 6-15b. If f is Riemann integrable on $[a, b]$, then $|F|$ is Riemann integrable on $[a, b]$ and $\displaystyle\left|\int_a^b f\right| \leq \int_a^b |f|$.

Corollary 6-15c. If f is a bounded Riemann integrable function on $[a,b]$ and $m \le f(x) \le M$ for $x \in [a,b]$ then

$$m(b-a) \le \int_a^b f \le M(b-a).$$

Theorem 6-16. Mean Value Theorem for Integrals.

Suppose f is continuous on $[a,b]$. Then there exists $c \in [a,b]$ such that

$$\int_a^b f = f(c)(b-a).$$

Theorem 6-17. Let f be bounded and Riemann integrable on $[a,b]$ and let $F(x) = \int_a^x f$ for $x \in [a,b]$. Then F is continuous on $[a,b]$.

Theorem 6-18. Let f be a continuous function on $[a,b]$ and let $F(x) = \int_a^x f$ for $x \in (a,b)$. Then F is differentiable at x and $F'(x) = f(x)$.

Theorem 6-19. Fundamental Theorem of Calculus.

Suppose f is a bounded Riemann integrable function on $[a,b]$ and suppose F is defined on $[a,b]$ such that

(i) F is *continuous* on $[a,b]$ and differentiable on (a,b).

(ii) $F'(x) = f(x)$ for all $x \in (a,b)$.

Then $\int_a^b f = F(b) - F(a)$.

Note. An example of a function f satisfying the hypotheses of the Fundamental Theorem of Calculus which is not continuous is

$$f(x) = \begin{cases} \frac{1}{2} + 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), & x \neq 0 \\ \frac{1}{2}, & x = 0. \end{cases}$$

Then f is not continuous at $x = 0$ and

$$F(x) = \begin{cases} \frac{x}{2} + x^2\sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is an antiderivative of f (that is, $F'(x) = f(x)$ for all $x \in \mathbb{R}$). Also notice that F is continuous (as guaranteed by Theorem 6-17). This example is from Ken Kubota's website `http://www.msc.uky.edu/ken/ma570/homework/hw8/html/ch2e.htm`, accessed spring 2003.

Corollary 6-19. Integration by Parts.

Suppose f and g have continuous derivatives on $[a, b]$. Then $\int_a^b fg'$ and $\int_a^b gf'$ exist and

$$\int_a^b fg' = (fg)|_a^b - \int_a^b gf'.$$

Theorem 6-20. Change of Variables.

Suppose g' is continuous on $[a, b]$, and that f is continuous on $g([a, b])$. Then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Definition. An integral is *improper* if either the function is unbounded or the interval of integration is unbounded.

Definition. Suppose f is defined on $[a, \infty)$ and Riemann integrable on $[a, t]$ for all $t \in (a, \infty)$. The *improper Riemann integral* of f on $[a, \infty)$ is

$$\int_a^\infty f = \lim_{t \to \infty} \int_a^t f,$$

provided the limit exists. If the limit exists as a finite number, the integral is said to *converge*. If the limit is $\pm\infty$, the limit *diverges to* $\pm\infty$. Otherwise, the integral is *divergent*. If f is defined on $(-\infty, \infty)$ and Riemann integrable on $[-t, a]$ and $[a, t]$ for all t and some a, then

$$\int_{-\infty}^\infty f = \int_{-\infty}^a f + \int_a^\infty f.$$

Definition. Suppose f is defined on $(a, b]$ and Riemann integrable on $[t, b]$ for all $t \in (a, b)$ and $\lim\limits_{t \downarrow a} f(t) = \pm\infty$. Then define

$$\int_a^b f = \lim_{r \downarrow a} \int_t^b f.$$

The integral is said to *converge*, *diverge to* $\pm\infty$, or be *divergent* as above.

Definition. For $x > 0$ define

$$\ln(x) = \int_1^x \frac{1}{t}\, dt.$$

Theorem.

(i) $\ln(1) = 0$.

(ii) $\ln(ab) = \ln(a) + \ln(b)$ for $a, b > 0$.

(iii) $\ln(a/b) = \ln(a) - \ln(b)$ for $a, b > 0$.

(iv) $\ln(a^r) = r\ln(a)$ for $a > 0$.

Theorem. $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^{n-1}} \geq 1 + (n-1)\frac{1}{2}$.

Theorem. $\lim_{n\to\infty} \ln(n) = \infty$.

Corollary. $\lim_{x\downarrow 0} \ln(x) = -\infty$.

Definition. Let e be the unique real number such that $\ln(e) = 1$.

Note. We have

$$\frac{d}{dx}[\ln x]\bigg|_{x=1} = \lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h} = \lim_{h\to 0}\frac{\ln(1+h)}{h} = \lim_{h\to 0}\ln(1+h)^{1/h} = \frac{1}{(1)} = 1.$$

So

$$\lim_{h\to 0}\ln(1+h)^{1/h} = \ln\lim_{h\to 0}(1+h)^{1/h} = 1$$

and

$$\lim_{h\to 0}\ln(1+h)^{1/h} = e \text{ or } \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Note. We have $\ln(e^x) = x\ln(e) = x$ for all $x \in \mathbb{R}$. Denote the inverse of $\ln(x)$ as e^x. Then:

(i) $e^0 = 1$.

(ii) $e^{a+b} = e^a e^b$.

(iii) $e^{a-b} = e^a/e^b$.

(iv) $e^{ax}(e^a)^x$.

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