6.3. The Riemann-Stieltjes Integral.

Note. In this section we define the Riemann-Stieltjes integral of function f with respect to function g. When $g(x) = x$, this reduces to the Riemann integral of f.

Definition. Let f be bounded on $[a, b]$, let g be nondecreasing on $[a, b]$, and let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$. The *upper (lower) Riemann-Stieltjes sum* of f with respect to g and with respect to P is

$$\overline{S}(f; g; P) = \sum_{i=1}^{n} M_i(f)[g(x_i) - g(x_{i-1})] = \sum_{i=1}^{n} M_i(f)\Delta g(x_i)$$

$$\left(\underline{S}(f; g; P) = \sum_{i=1}^{n} m_i(f)[g(x_i) - g(x_{i-1})] = \sum_{i=1}^{n} m_i(f)\Delta g(x_i)\right).$$

Note. Since g is nondecreasing, $\Delta g(x_i) \geq 0$.

Note. If $g(x) = x$ then $\Delta g(x_i) = \Delta x_i$ and Riemann-Stieltjes integrals will reduce to Riemann integrals.

Note. Theorem 6-21 gives relationships between refinements of a partition, compares upper and lower Riemann-Stieltjes sums, and defines $\overline{s}(f; g)$ and $\underline{s}(f; g)$. We say f is integrable with respect to g on $[a, b]$ is $\overline{s}(f; g) = \underline{s}(f; g)$.

Theorem 6-22. Riemann Condition for Integrability.

The function f is integrable with respect to g on $[a, b]$ if and only if for all $\varepsilon > 0$ there exists a partition $P(\varepsilon)$ of $[a, b]$ such that

$$|\overline{s}(f; g; P(\varepsilon)) - \underline{s}(f; g; P(\varepsilon))| < \varepsilon.$$

Example. Let

$$f(x) = \begin{cases} 0, & x \in [0, 1] \\ 1, & x \in (1, 2] \end{cases}$$

and

$$g(x) = \begin{cases} 0, & x \in [0, 1) \\ 1, & x \in [1, 2]. \end{cases}$$

Let P be a partition $\{x_0, x_1, x_2, \ldots, x_n\}$ of $[0, 2]$ with $x_j = 1$. Then

$$\begin{aligned} \overline{S}(f; g; P) &= \sum_{i=1}^{n} M_i(f)[g(x_i) - g(x_{i-1})] \\ &= M_i(f)[g(x_j) - g(x_{j-1})] \\ &= M_j(f)[1 - 0] = M_j(f) = 0 \end{aligned}$$

and

$$\begin{aligned} \underline{S}(f; g; P) &= \sum_{i=1}^{n} m_i(f)[g(x_i) - g(x_{i-1})] \\ &= m_i(f)[g(x_j) - g(x_{j-1})] \\ &= m_j(f)[1 - 0] = m_j(f) = 0. \end{aligned}$$

So, $\displaystyle\int_0^2 f\, dg = 0$.

Note. In the previous example, if 1 is *not* an element of P, then $\underline{S}(f;g;P) = 0$ and $\overline{S}(f;g;P) = 1$. So even through $\|P\|$ is small, $\underline{s}$ and $\overline{s}$ are not "close."

Theorem 6-23. If f is continuous on $[a,b]$, then f is integrable with respect to g.

Note. The following result gives a relationship between Riemann integrals and Riemann-Stieltjes integrals with respect to an increasing function.

Theorem 6-24. Suppose f is Riemann integrable on $[a,b]$ and g is an increasing function on $[a,b]$ such that g' is defined and Riemann integrable on $[a,b]$. Then f is integrable with respect to g on $[a,b]$, fg' in Riemann integrable on $[a,b]$ and

$$\int_a^b f\,dg = \int_a^b f(x)g'(x)\,dx.$$

Note. The following result shows that we have linearity with respect to both integrands and the "integrator" function.

Theorem 6-25.

(i) Suppose f_1 and f_2 are integrable with respect to g on $[a,b]$. Then $\alpha f_1 + \beta f_2$ is integrable with respect to g on $[a,b]$ and

$$\int_a^b (\alpha f_1 + \beta f_2)\,dg = \alpha \int_1^b f_1\,dg + \beta \int_a^b f_2\,dg.$$

(ii) Suppose f is integrable with respect to g_1 and g_2 on $[a,b]$. Then f is integrable with respect to $\alpha g_1 + \beta g_2$ (where α, β are nonnegative) on $[a,b]$ and

$$\int_a^b f\, d(\alpha g_1 + \beta g_2) = \alpha \int_a^b f\, dg_1 + \beta \int_a^b f\, dg_2.$$

Note. The following result shows how a Riemann-Stieltjes integral behaves when g has a jump discontinuity.

Theorem 6-26. Let f and g be such that g is increasing on $[a,b]$. Suppose for some $c \in (a,b)$:

(i) fg' has an antiderivative $F(x)$ on $[a,c)$ and an antiderivative $G(x)$ on $(c,b]$, and

$$\lim_{x\uparrow c} F(x) \text{ and } \lim_{x\downarrow c} G(x) \text{ exist.}$$

(ii) g has a jump discontinuity at c and f is continuous at c.

Then

$$\int_a^b f\, dg = \lim_{x\uparrow c}[F(x) - F(a)] + f(c)\left[\lim_{x\downarrow c} g(x) - \lim_{x\uparrow c} g(x)\right] + \lim_{x\downarrow c}[G(b) - G(x)].$$

Corollary 6-26. Theorem 6-26 holds for a finite number of discontinuities (by induction).

Problem 6-2-2(a). Consider

$$f(x) = \begin{cases} 2x & \text{for } x \in [0,1) \\ 2x^3 & \text{for } x \in (1,3/2] \\ e^x & \text{for } x \in (3/2,3] \end{cases} \quad \text{and } g(x) = \begin{cases} 0 & \text{for } x \in [0,1) \\ 1 & \text{for } x \in [1,2] \\ 3 & \text{for } x \in (2,3] \end{cases}$$

Then

$$\int_0^3 f\,dg = \int_0^1 f\,dg + f(1)\left[\lim_{x\downarrow 1} g(x) - \lim_{x\uparrow 1} g(x)\right] + \int_1^2 f\,dg$$

$$+f(2)\left[\lim_{x\downarrow 2} g(x) - \lim_{x\uparrow 2} g(x)\right] + \int_2^3 f\,dg = 0 + 2[1-0] + 0 + e^2[3-1] + 0 = 2 + 2e^2.$$

Definition. Suppose g is an increasing function and suppose f is Reimann-Stieltjes integrable with respect to g. Then define

$$\int_a^b f\,dg = -\int_a^b f\,d(-g).$$

Note. We can now deal with any integrator function g which can be written as $g = g_1 - g_2$ where both g_1 and g_2 are nondecreasing. Such a function as g is said to be of *bounded variation.*

Theorem 6-29. Integration by Parts.

Suppose f and g are increasing on $[1,b]$. Then f is Riemann-Stieltjes integrable with respect to g if and only if g is Riemann-Stieltjes integrable with respect to f. In this case,

$$\int_a^b f\,dg = f(b)g(b) - f(a)g(a) - \int_a^b g\,df.$$

Corollary 6-29(b). If f is monotone and g is continuous on $[a,b]$ then f is Riemann-Stieltjes integrable with respect to g on $[a,b]$.

Example. An interesting application of Riemann-Stieltjes integration occurs in probability theory. Consider a regular 6-sided die and the function

$$g(x) = \begin{cases} 0 & \text{for } x \in (-\infty, 1) \\ 1/6 & \text{for } x \in [1,2) \\ 2/6 & \text{for } x \in [2,3) \\ 3/6 & \text{for } x \in [3,4) \\ 4/6 & \text{for } x \in [4,5) \\ 5/6 & \text{for } x \in [5,6) \\ 1 & \text{for } x \in [6,\infty). \end{cases}$$

Let x represent the outcome of the throw of the die. We can compute the expected value of x as $\int_{-\infty}^{\infty} x\,dg$:

$$\int_{-\infty}^{\infty} x\,dg = \int_{-\infty}^{1} x\,dg + (1)\left[\lim_{x\downarrow 1} g(x) - \lim_{x\uparrow 1} g(x)\right] + \int_{1}^{2} x\,dg + (2)\left[\lim_{x\downarrow 2} g(x) - \lim_{x\uparrow 2} g(x)\right]$$
$$+ \int_{2}^{3} x\,dg + (3)\left[\lim_{x\downarrow 3} g(x) - \lim_{x\uparrow 3} g(x)\right] + \int_{3}^{4} x\,dg + (4)\left[\lim_{x\downarrow 4} g(x) - \lim_{x\uparrow 4} g(x)\right]$$
$$+ \int_{4}^{5} x\,dg + (5)\left[\lim_{x\downarrow 5} g(x) - \lim_{x\uparrow 5} g(x)\right] + \int_{5}^{6} x\,dg + (6)\left[\lim_{x\downarrow 6} g(x) - \lim_{x\uparrow 6} g(x)\right] + \int_{6}^{\infty} x\,dg$$
$$= 0 + (1)[1/6] + (2)[1/6] + (3)[1/6] + (4)[1/6] + (5)[1/6] + (6)[1/6] + 0 = 7/2.$$

Notice also that $\int_{-\infty}^{\infty} dg = 1$, as we would need from a probability distribution. So we can translate simple probability questions in the discrete setting which use summations into the continuous setting where summation is replaced with Riemann-Stieltjes integration. This allows us to apply the theory of integration even in the setting of discrete probability problems.

Example. The Dirac-Delta Distribution, $\delta(x)$, which you might encounter in a physics class is sometimes described (well, mis-described) as having the following properties: $\delta(x) = 0$ for $x \neq 0$, $\delta(x) = \infty$ for $x = 0$, and $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$. As you will see in a graduate level analysis class, there is no function satisfying these properties (See Problem 4.17 of Section 4.3 on "Measurable Nonnegative Functions" in Royden and Fitzpatrick's *Real Analysis*, 4th Edition—this is the book used in ETSU's Real Analysis 1 and 2 [MATH 5210/5220] classes). However, this can be dealt with using Riemann-Stieltjes integration and the function

$$g(x) = \begin{cases} 0 & \text{for } x \in (-\infty, 0) \\ 1 & \text{for } x \in [0, \infty). \end{cases}$$

Then the derivative of g is 0 if $n \neq 0$. The definition of limit gives that the derivative of g is ∞ at 0. Also, we have the Riemann-Stieltjes integral

$$\int_{-\infty}^{\infty} dg = \int_{-\infty}^{0} dg + (1)\left[\lim_{x\downarrow 0} g(x) - \lim_{x\uparrow 0} g(x)\right] + \int_{0}^{\infty} dg = 0 + (1)[1] + 0 = 1.$$

So the derivative of $g(x)$ has the values of $\delta(x)$ given above and the Riemann-Stieltjes integral of $f(x) = 1$ with respect to g satisfies the integral property of $\delta(x)$ given above. The Dirac Delta Distribution is used to locate point charges in electricity and magnetism. If f is a function defined on all of $\mathbb{R}$, then we can use the Riemann-Stieltjes integral to determine the value of f at a specific point (say $x = x_0$):

$$\int_{-\infty}^{\infty} f(x)\, dg(x - x_0) = f(x_0)\left[\lim_{x\downarrow x_0} g(x - x_0) - \lim_{x\uparrow x_0} g(x - x_0)\right] = f(x_0)[1] = f(x_0).$$