Chapter 7. Series of Real Numbers.

7.1. Tests for Convergence.

Note. In this section we define "series" and present several results concerning the convergence and divergence of series which will be familiar from Calculus 2.

Definition. A *series* $\sum_{n=1}^{\infty} a_n$ is the ordered pair of sequences $(\{a_n\}, \{S_n\})$ where $S_n = \sum_{i=1}^{n} a_i$. The sequence $\{S_n\}$ is the sequence of *partial sums* for the series. The numbers a_n are the *terms* of the series.

Definition. The series $\sum a_n$ *converges to* L if the sequence of partial sums $\{S_n\}$ converges to L. The series *diverges* if $\{S_n\}$ diverges and the series *diverges to* ∞ *or* $-\infty$ if $\{S_n\}$ diverges to ∞ or $-\infty$.

Note. The previous definition puts all the "ε/δ work" for series back on sequences. So we expect the convergence/divergence of series to behave similarly to the way sequences behave. In fact, the following theorem follows immediately from Theorem 2-4 which concerns linear combinations of sequences.

Theorem 7-1. Suppose $\sum a_n$ converges to a and $\sum b_n$ converges to b. Then for $\alpha, \beta \in \mathbb{R}$, $\sum(\alpha a_n + \beta b_n)$ converges to $(\alpha a + \beta b)$.

Theorem 7-2. The convergence or divergence of a series is not affected by the addition of a finite number of terms to the series.

Theorem 7-3. Test for Divergence.
If $\lim a_n \neq 0$ then $\sum a_n$ diverges.

Note. The Test for Divergence comes with the standard warning: It is a test for *divergence* and says nothing about convergence of a series. That is, we may have the terms a_n go to 0, yet the series may not converge.

Definition. A *geometric series* $\sum a_n$ is a series of the form

$$a + ar + ar^2 + ar^3 + \cdots.$$

Theorem 7-4. Let $a + ar + ar^2 + ar^3 + \cdots$ be a geometric series. If $a = 0$, the series converges to 0. If $a \neq 0$, the series converges to $\frac{1}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.

Note. If all the terms of a series are positive, then the partial sums form a monotone sequence.

Theorem 7-5. Let $\sum a_n$ be a "positive term series." Then $\sum a_n$ either converges or diverges to ∞.

Theorem 7-6. Comparison Test. Let $\sum a_n$ and $\sum b_n$ be positive term series with $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then

(a) If $\sum b_n$ converges then $\sum a_n$ converges.

(b) If $\sum a_n$ diverges to ∞ then $\sum b_n$ diverges to ∞.

Note. The down side of the Comparison Test is that we can only apply it to a given series if we can find another series for which (1) we know how the new series behaves, and (2) the new series compares "favorably" to the given series. For example, if we find a series $\sum b_n$ which diverges to ∞ and for which $a_n \leq b_n$, then this tells us nothing about the series $\sum a_n$. The following two results are easier to use in the sense that they compare a series to itself in order to test convergence.

Theorem 7-7. The Ratio Test.

Let $\sum a_n$ be a series of positive terms. Let

$$R = \overline{\lim}\frac{a_{n+1}}{a_n} \text{ and } r = \underline{\lim}\frac{a_{n+1}}{a_n}.$$

The series converges if $R < 1$ and diverges if $r > 1$.

Note. The version of the Ratio Test presented here is more robust that that presented in Calculus 2 in that this version does not require $\lim \frac{a_{n+1}}{a_n}$ to exist, since this version uses $\overline{\lim}$ and $\underline{\lim}$.

Theorem 7-8. The Root Test.

Let $\sum a_n$ be a series of nonnegative terms and let $\rho = \overline{\lim}(a_n)^{1/n}$. The series $\sum a_n$ converges if $\rho < 1$ and diverges if $\rho > 1$.

Note. In Exercise 7-1-19 it is shown that if the Ratio Test can be used to find the convergence or divergence of a series, then the Root Test will do the same for that particular series.

Theorem 7-9. The Integral Test.

Let $\sum a_n$ be a series of positive numbers with $a_1 \geq a_2 \geq a_3 \geq \cdots$. Let $f(x)$ be a nonincreasing continuous function of $(0, \infty)$ such that $f(n) = a_n$ for each positive integer n. Then

$$S_n = a_1 + a_2 + \cdots a_n \geq \int_1^n f(x)\,dx \text{ and } S_n - a_1 = a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)\,dx$$

so that the series $\sum a_n$ converges if and only if the improper integral $\int_1^\infty f(x)\,dx$ converges. If the series converges, then

$$\sum a_n \geq \int_1^\infty f(x)\,dx \geq \sum a_n - a_1.$$

Thus, if the integral test applies, we have bounds on the number to which a series converges.

Note. The down side of the Integral Test is that it requires us to evaluate a definite integral, and this can be difficult. But one important application is given in the following corollary.

Corollary 7-9. p-series Test.

The series $\sum 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$.

Definition. The divergent series $\sum 1/n$ is the *harmonic series.*

Note. The harmonic series is a positive term series that satisfies the condition $\lim a_n = 0$, yet it diverges. This example shows that there is no converse of the Test for Divergence.

Revised: 12/29/2014