7.2. Operations Involving Series.

Note. In this section we study absolutely and conditionally convergent sequences. We introduce alternating series and the Alternating Series Estimation Theorem. Most of this material is covered in Calculus 2 (MATH 1920).

Definition. A series $\sum a_n$ is *absolutely convergent* if $\sum |a_n|$ is convergent. If $\sum a_n$ converges and $\sum |a_n|$ diverges, then $\sum a_n$ is *conditionally convergent.*

Theorem 7-10. If a series is absolutely convergent then it is convergent.

Corollary 7-7. For $\sum a_n$, with $a_n \neq 0$ for all $n \in \mathbb{N}$, let

$$R = \overline{\lim} \left| \frac{a_{n+1}}{a_n} \right| \text{ and } r = \underline{\lim} \left| \frac{a_{n+1}}{a_n} \right|.$$

Then $\sum a_n$ converges absolutely if $R < 1$ and diverges if $r > 1$.

Corollary 7-8. For $\sum a_n$, let $\rho = \overline{\lim}(|a_n|)^{1/n}$. Then $\sum a_n$ converges absolutely if $\rho < 1$ and diverges if $\rho > 1$.

Theorem 7-11. Let $\sum a_n$ be conditionally convergent. Choose the positive terms of $\sum a_n$ and produce $\sum b_n$ and choose the negative terms of $\sum a_n$ and produce $\sum c_n$. Then $\sum b_n$ diverges to ∞ and $\sum c_n$ diverges to $-\infty$.

Definition. Let f be a one to one and onto function from $\mathbb{N} \to \mathbb{N}$. Let $\sum a_n$ be a series and define $\sum b_n$ where $b_n = a_{f(n)}$. Then $\sum b_n$ is a *rearrangement* of $\sum a_n$.

Theorem 7-12. Let $\sum a_n$ be conditionally convergent. Then given any number L (finite or infinite), there is a rearrangement of $\sum a_n$ that converges to L.

Note. You should find Theorem 7-12 shocking! The warning to take away from this result is that you must be very careful when manipulating series. In particular, conditionally convergent can be ill-behaved.

Example. We can rearrange the alternating harmonic series to converge to 1. We start with the first term 1/1 and then subtract 1/2. Next we add 1/3 and 1/5, which brings the total back to 1 or above. Then we add consecutive negative terms until the total is less than 1. We continue in this manner: When the sum is less than 1, add positive terms until the total is 1 or more, then subtract (add negative) terms until the total is again less than 1. This process can be continued indefinitely. Because both the odd numbered terms and the even-numbered terms of the original series approach 0 as $n \to \infty$, the amount by which our partial sums exceed 1 or fall below it approaches 0. So the new series converges to 1. The rearranged series starts like this:

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} - \frac{1}{6} + \frac{1}{11} + \frac{1}{13} - \frac{1}{8} + \frac{1}{15} + \frac{1}{17} - \frac{1}{10} + \frac{1}{19}$$
$$+ \frac{1}{21} - \frac{1}{12} + \frac{1}{23} + \frac{1}{25} - \frac{1}{14} + \frac{1}{27} - \frac{1}{16} + \cdots.$$

Theorem 7-13. Suppose $\sum a_n$ converges to L and $\sum b_n$ is obtained from $\sum a_n$ by inserting parentheses. Then $\sum b_n$ converges to L.

Theorem 7-14. Let $\sum a_n$ be absolutely convergent. Let $\sum a_n'$ be defined by

$$a_n' = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0. \end{cases}$$

and $\sum a_n''$ be defined by

$$a_n'' = \begin{cases} a_n & \text{if } a_n < 0 \\ 0 & \text{if } a_n \geq 0. \end{cases}$$

Then $\sum a_n'$ and $\sum a_n''$ are absolutely convergent and $\sum a_n = \sum a_n' + \sum a_n''$.

Theorem 7-15. Let $\sum a_n$ be absolutely convergent and let $\sum a_n'$ and $\sum a_n''$ be as in Theorem 7-14. If $\sum a_n' = P$ and $\sum a_n'' = M$ then any rearrangement of $\sum a_n$ converges to $P + M$.

Definition. If $a_n > 0$ for every $n \in \mathbb{N}$, then the series $\sum(-1)^n a_n$ and $\sum(-1)^{n+1} a_n$ are called *alternating series.*

Theorem 7-16. Alternating Series Test.

Let $\sum(-1)^{n+1} a_n$ be alternating such that

(i) $a_n \geq a_{n+1} \geq 0$ for all $n \in \mathbb{N}$,

(ii) $\lim a_n = 0$.

Then $\sum(-1)^n a_n$ and $\sum(-1)^{n+1} a_n$ converges.

Example. The alternating p-series $\sum \frac{(-1)^{n-1}}{n^p}$ converges for all $p > 0$. We have seen that for $p > 1$ the series, in fact, converges absolutely. However, since for $0 < p \leq 1$ the regular p-series diverges, we see that the alternating p-series are conditionally convergent for these values of p.

Corollary 7-16.

(a) If $\sum(-1)^{n+1}a_n$ is alternating and satisfies the hypotheses of Theorem 7-16 and converges to L, then $L < a_1$.

(b) As above, if $\{S_n\}$ is the sequence of partial sums, then $|L - S_n| < |a_{n+1}|$.

Note. Part (b) of Corollary 7-16 is the "Alternating Series Estimation Theorem" and shows that the sum of the first n terms of an alternating series (which satisfies the hypotheses of Theorem 7-16)) approximates the sum of the series with an error of $|a_{n+1}|$.

Example. If we sum the first 99 terms of the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, then we are within $|a_{100}| = 0.01$ of the actual sum. By the way, this gives some indication of how slowly the alternating harmonic series converges.

Definition. Let $\sum a_n$ and $\sum b_n$ be series. The *Cauchy product* of these is $\sum c_n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Note. It is easy to show that the Cauchy product of two absolutely convergent series is absolutely convergent. But there is a stronger result.

Theorem 7-17. Merten's Theorem.

If $\sum a_n$ is absolutely convergent and $\sum b_n$ is convergent, then the Cauchy product of these two series converges to $(\sum a_n)(\sum b_n)$.

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