Chapter 8. Sequences and Series of Functions.

8.1. Sequences of Functions.

Note. In this section we define the pointwise and uniform limit of a sequence of functions. We show how the limit function relates to the functions in the sequence in terms of continuity and integrability. We prove the second most important result of Analysis 2 in Theorem 8-3!

Note. For a given sequence of functions f_n, if for all $x \in E$ the sequence of numbers $\{f_n(x)\}$ converges, then we can define $f(x) = \lim_{n\to\infty} f_n(x)$ with domain E. Are there properties of the f_n functions which are shared by the limit function f?

Question 1. If f_n is continuous for all $n \in \mathbb{N}$, is $\lim_{n\to\infty} f_n$ continuous?
Answer 1. NO! Let $f_n(x) = x^n$ for $x \in E = [0,1]$.

Question 2. If $\int_a^b f_n(x)\,dx = 1$ for all $n \in \mathbb{N}$, is $\int_a^b f(x)\,dx = 1$?
Answer 2. NO! Consider

$$f_n(x) = \begin{cases} n^2 x & \text{if } x \in [0, 1/n] \\ 2n - n^2 x & \text{if } x \in (1/n, 2/n] \\ 0 & \text{if } x \in (2/n, \infty). \end{cases}$$

Then $\int_0^1 f_n(x)\,dx = 1$ for all $n \in \mathbb{N}$, but $\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$.

Note. In fact, a limit of integrable functions need not be integrable. Let $q_1, q_2, \ldots$ be an enumeration of $\mathbb{Q}$ and define

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{q_1, q_2, \ldots, q_n\} \\ 0 & \text{if } x \in \mathbb{R} \setminus \{q_1, q_2, \ldots, q_n\}. \end{cases}$$

Then $f = \lim_{n\to\infty} f_n$ is the *Dirichlet function*, given by the formula

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Notice that each f_n is integrable, but f is not integrable. If a series is absolutely convergent then it is convergent.

Definition. Let f_n be defined on E and suppose $\{f_n(x)\}$ is a convergent sequence of numbers for each $x \in E$. Then for all $x \in E$, define $f(x) = \lim_{n\to\infty} f_n(x)$. The function f is the *pointwise limit* of the f_n's.

Note. We see from the questions above that the pointwise limit is not sufficient to preserve properties of the f_n in the function f. We need a stronger version of convergence for a sequence of functions.

Definition. Let f_n be defined on E and suppose $\{f_n(x)\}$ converges. Define for all $x \in E$, $f(x) = \lim_{n\to\infty} f_n(x)$. Then f is the *uniform limit* of $\{f_n\}$ if

$$\text{for all} \varepsilon > 0, \text{ there exists } N(\varepsilon) \in \mathbb{N} \text{ such that}$$

$$\text{if } n > N(\varepsilon) \text{ then } |f_n(x) - f(x)| < \varepsilon \text{ for all } x \in E.$$

Sequence $\{f_n\}$ is said to *converge uniformly* to f on set E.

Note. The "uniform" in uniform limit is based on the fact that the same $N(\varepsilon) \in \mathbb{N}$ "works" *for all* $x \in E$.

Theorem 8-1. Suppose $\{f_n\}$ converges pointwise to f on E. Let

$$M_n = \sup_{x \in E} |f_n(x) - f(x)|.$$

Then $\{f_n\}$ converges uniformly to f on E if and only if $\lim_{n\to\infty} M_n = 0$.

Note. Notice that for $f_n(x) = x^n$ on $E = [0,1]$, we have $M_n = \sup_{x \in E} |f_x(x) - f(x)| = 1$, and we see that $\{f_n\}$ does not converge uniformly on $[0,1]$. Notice that the (pointwise) limit function is the discontinuous function

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1. \end{cases}$$

Theorem 8-2. Let f be the uniform limit of a sequence of continuous functions $\{f_n\}$. Then f is continuous.

Note. We now see why the answer to Question 1 is "no." Pointwise convergence is not sufficient force a limit of continuous functions to be continuous—but uniform convergence is.

Note. The following theorem is **the second most important result in Analysis 2** (after the Riemann-Lebesgue Theorem)!!! It tells us when the limit of the (Riemann) integrals of a sequence of functions is the integral of the limit function.

Theorem 8-3. Suppose $\{f_n\}$ is a sequence of Riemann integrable functions on $[a,b]$. If $\{f_n\}$ converges uniformly to f on $[a,b]$, then f is Riemann integrable on $[a,b]$ and

$$\lim_{n\to\infty}\left(\int_a^b f_n(x)\,dx\right) = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right)\,dx = \int_a^b f(x)\,dx.$$

Proof. First, we show that the limit function f is integrable. Let $\varepsilon > 0$. Since $\{f_n\} \to f$ uniformly on E, there exists $N \in \mathbb{N}$ such that for all $x \in [a,b]$ we have

$$|f(x) - f_N(x)| < \frac{\varepsilon}{6(b-a)}.$$

Since f_N is Riemann integrable, there is a partition $P(\varepsilon)$ of $[a,n]$ such that $\overline{S}(f_N;P) - \underline{S}(f_N : P) < \varepsilon/3$ by the Riemann Condition for Integrability (Theorem 6-4). In Exercise 8-1-11 it is shown that

$$|f(x) - f_N(x)| < \frac{\varepsilon}{6(b-a)} \text{ implies } |M_i(f) - M_i(f_N)| < \frac{\varepsilon}{3(b-a)}$$
$$\text{and } |m_i(f) - m_i(f_N)| < \frac{\varepsilon}{3(b-a)}.$$

So

$$\begin{aligned}|\overline{S}(f;P) - \overline{S}(f_N;P)| &\le \sum_i |M_i(f) - M_i(f_N)|\Delta x_i \\ &< \frac{\varepsilon}{3(b-a)}\sum_i \Delta x_i = \frac{\varepsilon}{3(b-a)}(b-a) = \frac{\varepsilon}{3}.\end{aligned}$$

Also

$$|\underline{S}(f;P) - \underline{S}(f_N;P)| \le \sum_i |m_i(f) - m_i(f_N)|\Delta x_i < \frac{\varepsilon}{3}.$$

Therefore for this particular partition,

$$|\overline{S}(f;P) - \underline{S}(f_N;P)| \;=\; |\overline{S}(f;P) - \overline{S}(f_N;P) + \overline{S}(f;P)$$

$$
\begin{aligned}
& \quad -\underline{S}(f_N;P) + \underline{S}(f;P) - \underline{S}(f_N;P)| \\
\le & \ |\overline{S}(f;P) - \overline{S}(f_N;P)| + |\overline{S}(f;P) - \underline{S}(f_N;P)| \\
& \quad +|\underline{S}(f;P) - \underline{S}(f_N;P)| \\
< & \ \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$

So by the Riemann Condition for Integrability (Theorem 6-4), f is integrable.

Now for the value of the integral. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n > \mathbb{N}$ we have

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } x \in [a,b]$$

by the uniform convergence. Therefore, $f_n(x) - \varepsilon < f(x) < f_n(x) + \varepsilon$ for all $x \in [a,b]$. This implies

$$\int_a^b (f_n(x) - \varepsilon)\,dx < \int_a^b f(x)\,dx < \int_a^b (f_n(x)_\varepsilon)\,dx,$$

or

$$\int_a^b f_n(x)\,dx - \varepsilon(b-z) < \int_a^b f(x)\,dx < \int_a^b f_n(x)\,dx + \varepsilon(b-a).$$

Therefore

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| < \varepsilon(b-a)$$

for all $n < N$ and so the sequence $\left\{ \int_a^b f_n(x)\,dx \right\}$ converges to $\int_a^b f(x)\,dx$. ∎

Note. One of the main questions addressed in graduate level Real Analysis (MATH 5210-5220) is when it is that we can interchange the processes of limit and integration. Theorem 8-3 shows that for Riemann integration, we need uniform convergence. In graduate real analysis, we develop another type of integral (the Lebesgue integral) for which the condition of uniform convergence can be weakened, and yet limit and integration can still be interchanged. Such results are called "convergence theorems." For more details see my online notes for Real Analysis 1 (MATH 5210). In particular, see the results in Chapters 4 and 5.

Note. The fact that we consider integrals over closed and bounded intervals is necessary in Theorem 8-3, as shown in Example 8-7: Let

$$f_n(x) = \begin{cases} 1/n & \text{if } x \in [0,n] \\ 0 & \text{if } x > n. \end{cases}$$

Then $f_n \to 0$ uniformly on $[0,\infty)$ but

$$\int_0^\infty f_n(x)\,dx = 1 \neq \int_0^\infty 0\,dx = 0.$$

Note. We might expect that uniform convergence solved all of our problems and preserves all sorts of properties of the f_n's. However, Example 8-4 shows that uniform convergence is not sufficient to insure that a convergent sequence of differentiable functions has the expected derivative: Let $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ for $x \in [0,1]$. Then $lim_{n\to\infty} f_n(x) = 0$ (uniformly) on $[0,1]$. But $f_n'(x) = \sqrt{n}\cos(nx)$ and $\lim_{n\to\infty} f_n'(x)$ does not exist.

Theorem 8-4. Let $\{f_n\}$ be a sequence defined on set E. Then $\{f_n\}$ converges uniformly to f on E if and only if

$$\text{for all } \varepsilon > 0 \text{ there exists } N(\varepsilon) \in \mathbb{N} \text{ such that}$$

$$\text{if } x \in E \text{ and } m, n > N(\varepsilon) \text{ then } |f_n(x) - f_m(x)| < \varepsilon.$$

Theorem 8-5. Let $\{f_n\}$ be a sequence that converges uniformly to f on $[a, b]\backslash\{x_0\}$ where $x_0 \in [a, b]$. Suppose $\lim_{x\to x_0} f_n(x)$ exists for all $n \in \mathbb{N}$. Then

$$\lim_{x\to x_0}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty}\left(\lim_{x\to x_0} f_n(x)\right).$$

Theorem 8-6. Let $\{f_n\}$ be a sequence of functions which are differentiable on $[a, b]$. Suppose

(i) there exists $x_0 \in [a, b]$ where $\{f_n(x_0)\}$ converges,

(ii) $\{f_n'\}$ converges uniformly on $[a, b]$.

Then

(a) $\{f_n\}$ converges uniformly to f on $[a, b]$, and

(b) $f'(x) = \lim_{n\to\infty} f_0'(x)$ on (a, b).

Note. This may seem an odd time to address this, but we finally show that the derivative of x^r is rx^{r-1} for all $r \in \mathbb{R}$. Of course, we only need to establish this for r irrational.

Corollary. (Example 8-8) For r irrational, $\frac{d}{dx}[x^r] = rx^{r-1}$.

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