8.2. Series of Functions.

Note. In this section we introduce power series and explore the sets on which a power series may converge, converge uniformly, or diverge. We consider integrals and derivatives of power series.

Definition. For a series of functions $\sum f_n$, define the *partial sum* $S_n(x) = \sum_{i=1}^{n} f_i(x)$. If $\lim_{n\to\infty} S_n(x) = f(x)$, we say that the series $\sum f_n$ *converges to* f (on set E) and write $f = \sum f_n$. If $\{S_n\}$ converges uniformly to f on set E, we say $\sum f_n$ *converges uniformly* to f on E.

Theorem 8-7. Cauchy Criterion for Uniform Convergence.
The series $\sum f_n$ converges uniformly on E if and only if

$$\text{for all } \varepsilon > 0 \text{ there exists } N(\varepsilon) \in \mathbb{N} \text{ such that}$$
$$\left| \sum_{i=n+1}^{m} f_i(x) \right| < \varepsilon \text{ for all } x \in E.$$

Theorem 8-8. The Weierstrass M-Test.
Let $\{f_n\}$ be a sequence defined on E. Suppose for all $n \in \mathbb{N}$ there exists $M_n \in \mathbb{R}$ such that $|f_n(x)| \leq M_n$ for all $x \in E$. If $\sum M_n$ converges then $\sum f_n$ converges uniformly on E.

Note. The Weierstrass M-Test is not "if and only if." Exercise 8-2-2 considers the series $\sum f_n$ on $[0,1]$, where

$$f_n(x) = \begin{cases} 1/n & \text{if } x \in (1/2^n, 1/2^{n-1}] \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown that $\sum f_n$ converges uniformly, but that the Weierstrass M-test fails.

Corollary 8-2. If $\{f_n\}$ is a sequence of continuous functions on E and if $\sum f_n$ converges *uniformly* to f on E, then f is continuous on E.

Note. The following result will be useful when we express a function as a power series.

Corollary 8-3. Let $\{f_n\}$ be a sequence of integrable functions on $[a,b]$. If $\sum f_n$ converges uniformly to f on $[a,b]$ then $\int_a^b f(x)\,dx$ exists and

$$\int_a^b f(x)\,dx = \int_a^b \left(\sum_{n=1}^{\infty} f_n(x) \right) dx = \sum_{n=1}^{\infty} \left(\int_a^b f_n(x)\,dx \right).$$

Corollary 8-5. Let $\{f_n\}$ be a sequence of differentiable function on $[a, b]$ such that

(i) $\sum f_n(x_0)$ converges for some $x_0 \in [a, b]$, and

(ii) $\sum f_n'(x)$ converges uniformly to $h(x)$ on $[a, b]$.

Then

(a) $\sum f_n$ converges uniformly to f on $[a, b]$, and

(b) $\sum f_n'(x) = h(x) = f'(x)$ on (a, b).

Definition. Let $\{a_n\}$ be a sequence of real numbers. Then $\sum_{n=0}^{\infty} a_n(x - c)^n$ is a *power series* in $(x - c)$. The a_n are the *coefficients*.

Note. We are interested in the usual: convergence (for which x?), continuity, integrability, and differentiabilty.

Theorem 8-9. Let $\sum a_n x^n$ be a power series and let

$$\lambda = \overline{\lim}_{n\to\infty} \left(|a_n|^{1/n}\right) \text{ and } R = \frac{1}{\lambda}.$$

Then $\sum a_n x^n$

(a) converges absolutely if $|x| < R$,

(b) diverges if $|x| > R$, and

(c) if $0 < R < \infty$ then $\sum a_n x^n$ converges uniformly on $[-R+\varepsilon, R-\varepsilon]$ for all $\varepsilon > 0$.

Definition. The number R in the previous result is the *radius of convergence* of the power series $\sum a_n x^n$.

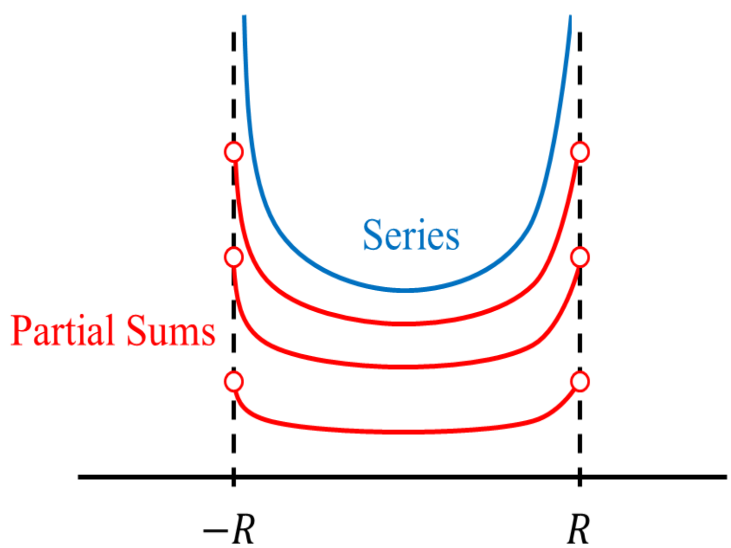
Theorem 8-10. Let $\sum a_n x^n$ be a power series with $a_n \neq 0$ for all $n \in \mathbb{N}$. Suppose $\lambda = \lim_{n\to\infty} |a_{n+1}/a_n|$ exists and let $R = 1/\lambda$. Then $\sum a_n x^n$

(a) converges absolutely if $|x| < R$,

(b) diverges if $|x| > R$, and

(c) converges uniformly on $[-R+\varepsilon, R-\varepsilon]$ for all $\varepsilon > 0$ if $0 < R < \infty$.

Note. Condition (c) does not imply uniform convergence on $(-R, R)$. Consider the following graph which might represent partial sums of a series.



Example. Find the x values for which the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ converges absolutely, converges conditionally, and diverges.

Corollary 8-10a. Let $\sum a_n x^n$ has radius of convergence R. If $x \in (-R, R)$, then the power series is continuous at x.

Note. The following result shows that we evaluate the definite integral of a power series in the obvious way. This is where Corollary 8-3 shows its strength.

Corollary 8-10b. Let $\sum a_n x^n$ have radius of convergence R and suppose $[a, b] \subset (-R, R)$. Then

$$\int_a^b \left(\sum_{n=0}^{\infty} a_n x^n \right) dx = \sum_{n=0}^{\infty} a_n \left(\int_a^b x^n \, dx \right).$$

Theorem 8-11. If $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then so does $\sum_{n=1}^{\infty} a_n n x^{n-1}$.

Note. The following result shows that we can differentiate a power series term by term.

Corollary 8-11. Let $\sum a_n x^n$ have radius of convergence $R > 0$. Then if $x \in (-R, R)$,

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} a_n x^n \right] = \sum_{n=1}^{\infty} a_n n x^{n-1}.$$

Theorem 8-12. Suppose $\sum a_n x^n$ and $\sum b_n x^n$ both converge on $(-R, R)$ for some $R > 0$. If $\sum a_n x^n = \sum b_n x^n$ on $(-R, R)$ then $a_n = b_n$ for all n.