8.3. Taylor Series.

Note. In this section we give necessary and sufficient conditions for a function to have a power series representation.

Definition. Function $f(x)$ defined on interval I is in class $C^n(I)$ if $f^{(n)}$ exists and is continuous for all $x \in I$. If $f \in C^n(I)$ for all $n \in \mathbb{N}$, then f is infinitely differentiable and $f \in C^\infty$.

Example. Consider

$$f(x) = \begin{cases} -x^2/2 & \text{if } x \in (-\infty, 0) \\ x^2/2 & \text{if } x \in [0, \infty). \end{cases}$$

We have $f'(x) = |x|$, so $f \in C^1(\mathbb{R})$ but $f \notin C^2(\mathbb{R})$.

Note. If $f \in C^\infty(I)$, $c \in I$, and $f(x) = \sum_{n=0}^\infty a_n(x-c)^n$, then we find that $a_n\frac{f^{(n)}(c)}{n!}$ and in fact

$$f(x) = \sum_{n=0}^\infty \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

Definition. In the above note, $f(x) = \sum_{n=0}^\infty \frac{f^{(n)}(c)}{n!}(x-c)^n$ is the *Taylor series* of f about $x = c$. If $c = 0$, this is called the *MacLaurin series* of f.

Definition. If there exists open interval I where $c \in I$ and $f(x) = \sum_{n=0}^\infty \frac{f^{(n)}(c)}{n!}(x-c)^n$ for all $x \in I$, then f is *analytic* at $x = c$.

Note. The study of analytic functions is central in complex analysis. You may encounter a different definition of "analytic" in the complex setting, but ultimately it will imply a power series representation. For more details, see some of the class notes for Complex Analysis 1 (MATH 5510):

`http://faculty.etsu.edu/gardnerr/5510/notes/III-2.pdf`

Note. Some familiar series are:

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \text{ for } |x| < 1 \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \text{ for } x \in \mathbb{R} \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \text{ for } x \in \mathbb{R} \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for } x \in \mathbb{R} \\
\ln x &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} \text{ for } x \in (0,2)
\end{aligned}$$

Note. You might think that if function f is infinitely differentiable, then it has a power series representation (after all, the converse of this as described above). However, there is a function $f \in C^\infty(\mathbb{R})$ where f is not analytic at $0 \in \mathbb{R}$. Consider

$$f(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0] \\ e^{-1/x^2} & \text{if } x \in (0, \infty). \end{cases}$$

We find that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$, so IF f has a MacLaurin series representation then $f \equiv 0$, a contradiction. So f is not analytic at $c = 0$.

Recall. Theorem 5-10, Taylor's Theorem.

Suppose $f \in C^{n+1}(I)$ where I is an open interval containing $[a, x]$. Then there exists $c \in (a, b)$ such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Definition. For $f \in C^n(I)$, define the *Taylor polynomial* of degree n for f at $x = a$ as

$$T_n(f; a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

and define the *remainder function*

$$R_n(x) = f(x) - T_n(f; a).$$

Theorem 8-13. Integral Form of the Remainder.

Suppose $f \in C^{n+1}(I)$ for interval I containing a. Then for $x \in I$,

$$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\, dt.$$

Corollary 8-13a. Suppose the hypotheses of Theorem 8-13 hold and that $m \leq f^{(n+1)}(t) \leq M$ on I. Then

$$m\frac{(x-a)^{n+1}}{(n+1)!} \leq R_n(x) \leq M\frac{(x-a)^{n+1}}{(n+1)!} \text{ if } x > a$$

and

$$m\frac{(x-a)^{n+1}}{(n+1)!}(-1)^{n+1} \leq R_n(x) \leq M\frac{(x-a)^{n+1}}{(n+1)!}(-1)^{n+1} \text{ if } x < a.$$

Note. Corollary 8-13a allows us to choose n such that $R_n(x)$ is "small." The following is more useful.

Corollary 8-13b. Lagrange Form of the Remainder.
Suppose $f \in C^n(I)$. For fixed $a \in I$, choose $x \in I$ with $x \neq a$. There exists ξ_n between x and z such that

$$f(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}+\frac{f^{(n)}(\xi_n)}{n!}(x-a)^n.$$

Example. Use Corollary 8-13b to numerically approximate $\sin(31^\circ)$ to the nearest 0.001.

Corollary 8-13c. Taylor Law of the Mean.
Suppose $f \in C^n(I)$ where I is an open interval containing $[a, b]$. Then there exists ξ_n with $a < \xi_n < b$ where

$$f(b) = f(a)+f'(a)(b-a)+\frac{f''(a)}{2!}(b-a)^2+\cdots+\frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1}+\frac{f^{(n)}(\xi_n)}{n!}(b-a)^n.$$

Note. A function will equal its Taylor series only when the remainder goes to 0. This is reflected in the following two results.

Theorem 8-14. Let $f \in C^\infty(I)$ and let a be an interior point of I. Then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \text{ if and only if } \lim_{n\to\infty} R_n(x) = 0.$$

Theorem 8-15. Let $f \in C^\infty((a,b))$ and $c \in (a,b)$. Suppose there is open interval I with $c \in I$ and some $A \in \mathbb{R}$ such that $|f^{(n)}(x)| < A^n$ for all $x \in I$ and for all $n \in \mathbb{N}$. Then For all $x \in I \cap (a,b)$ we have

$$f(x) = \sum_{n=0} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

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