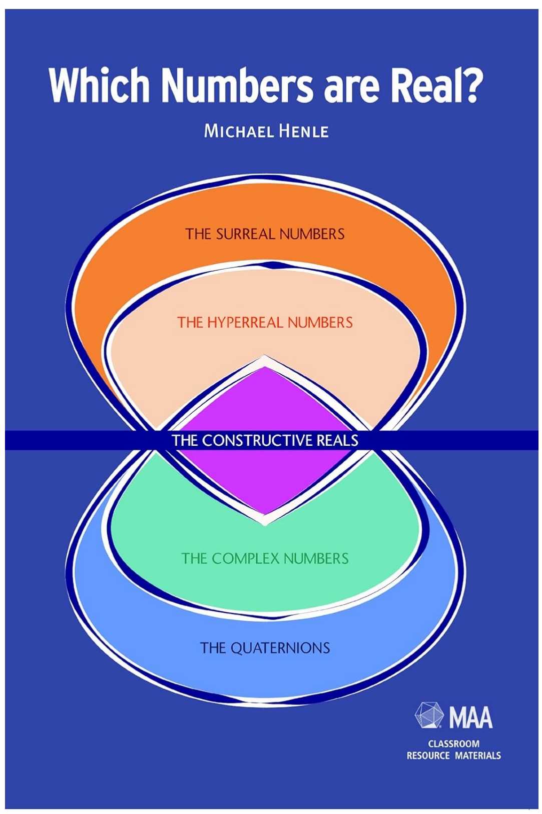
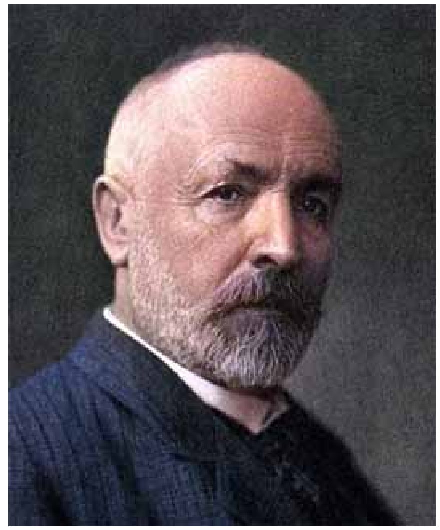
Supplement. The Real Numbers are the Unique Complete Ordered Field.

Note. In this supplement we prove that, up to isomorphism, there is only one complete ordered field. The source for this supplement is Michael Henle's *Which Numbers are Real?* (Mathematical Association of America, 2012). Section 2.3 from this source, "Uniqueness of the Reals," contains the main result of this supplement (in Theorem 2.3.3). We give a complete proof of this result, along with all necessary background definitions and results. In particular, we give a construction of the real numbers which starts with the rational numbers, $\mathbb{Q}$, and then defines a real number as an equivalence class of Cauchy sequences of rational numbers.



Note RU.A. The construction of the reals which we present is due to Georg Cantor (March 3, 1845–January 6, 1918). Cantor, largely famous for his work on cardinalities of sets and transfinite numbers, published his work in: Ueber die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen ("On the extension of a theorem from the theory of trigonometric series"), *Mathematische Annalen* **5**, 123–132 (1872). A related work concerning a construction of the real numbers is due to Richard Dedekind (October 6, 1831–February 12, 1916). Dedekind was working on his ideas (which are known today as *Dedekind cuts*) in the 1850s, but he did not put them in print until 1872 in: Stetickeit und irrationale zahlen ("Continuity and irrational numbers"). Dedekind states on his page 3 that he has just received Cantor's paper and judges "the axioms given in Section II of [Cantor's paper], aside from the form of presentation, agree with what I designate. . . as the essence of continuity." Dedekind's work is in print today as *Essays on the Theory of Numbers: I. Continuity and Irrational Numbers, II. The Nature and Meaning of Numbers*, Dover Publications (1963). It is also online on the Project Gutenberg webpage (accessed 11/14/2023). However, there does not seem to be a translation of Cantor's 1872 paper (that I can find, as of fall 2023). Dedekind considers a partitioning of the real number line into two sets, A and B, such that for any $a \in A$ and $b \in B$ we have $a < b$, $A \cap B = \varnothing$, and $A \cup B = \mathbb{R}$. Dedekind's Axiom of Completeness then states that, for any such two sets A and B (called a *Dedekind cut*, exactly one of the following holds: (1) there is a greatest number in set A, or (2) there is a least number in set B. This is described, somewhat informally, in my online notes for Calculus 1 on Appendix A.6. Theory of the Real Numbers. In this supplement, we consider Cantor's approach based on Cauchy sequenced of rational numbers.

Images are from the Richard Dedekind biography webpage (left) and the Georg Cantor biography webpage (right) of the MacTutor history of math website.

Note. We defined a Cauchy sequence of real numbers in Section 2.1. Sequences of Real Numbers. In Section 2.3. Bolzano-Weierstrass Theorem, we proved that a sequence of real numbers is Cauchy if and only if it is convergent (see Exercises 2.3.13 and 2.3.14). In order to introduce these ideas in an ordered field with positive set P, we need definitions of a convergent sequence and a Cauchy sequence that do not depend on "all real numbers $\varepsilon > 0$." For our purposes, the ordered field will be $\mathbb{Q}$. In these notes, the sequence denoted as $\{x_n\}_{n=1}^{\infty} = \{x_n\}$ in the Analysis 1 (MATH 4217/5217) notes, is denoted as $x = \{x(n)\}$ here.

Definition. (Limit & Cauchy in Ordered Field) An infinite sequence $\{x(n)\}$ from an ordered field with positive set P has *limit* L if, given any element $\varepsilon \in P$, there is a natural number $N(\varepsilon)$ such that for all $n > N(\varepsilon)$ we have $|x(n) - L| < \varepsilon$. The sequence $\{x(n)\}$ is a *Cauchy sequence* if, given any element $\varepsilon \in P$, there is a natural number $N(\varepsilon)$ such that for all $m, n > N(\varepsilon)$ we have $|x(n) - x(m)| < \varepsilon$.

Note. In much of what follows, we are taking the ordered field to be $\mathbb{Q}$, so that the "$\varepsilon \in P$" in the definition of limit and Cauchy sequence involves $\varepsilon \in \mathbb{Q}$ with $\varepsilon > 0$.

Definition. (Null) A *null* sequence of rational numbers is a convergent sequence whose limit is zero.

Note. We next give some properties of Cauchy sequences and null sequences of rational numbers. The proofs of parts (2) and (3) of Theorem 2.1.1 are to be given in Exercises 2.1.1 and 2.1.2 of Henle.

Theorem 2.1.1. Let $x = x\{x(n)\}$ and $y = \{y(n)\}$ be sequences of rational numbers.

(1) If x and y are Cauchy sequences, then so are $\{x(n) + y(n)\}$ and $\{x(n)y(n)\}$.

(2) If x and y are null sequences, then so are $\{x(n) + y(n)\}$ and $\{x(n)y(n)\}$.

(3) If x is a Cauchy sequence and y is a null sequence, then $\{x(n)y(n)\}$ is a null sequence.

Note. The next definition is fundamental to these notes. We will use it in Cantor's definition of real numbers.

Definition. (Equivalent) Two sequences of rational numbers $x = \{x(n)\}$ and $y = \{y(n)\}$ are *equivalent*, written $x \sim y$, if the difference $\{x(n) - y(n)\}$ is a null sequence. The *equivalence class* under $\sim$ containing sequence x is

$$S_x = \{y = \{y(n)\} \mid y \sim x\} \text{ (also denoted } \mathbf{x}).$$

The set of all equivalence classes is denoted $S/\sim$.

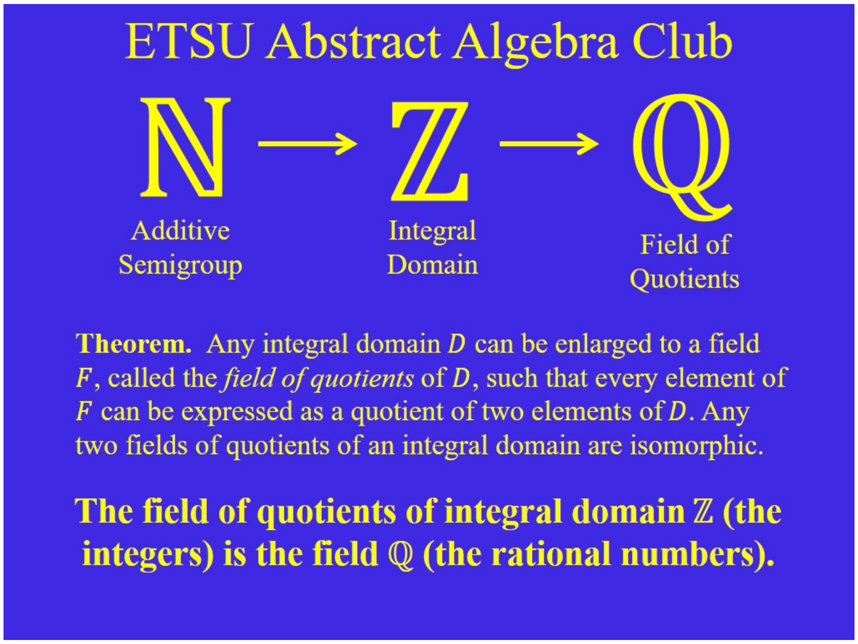
Theorem 2.1.2. Equivalence of sequences of rational numbers, $\sim$, is an equivalence relation.

Note. The proof of Theorem 2.1.2 is to be given in Exercise 2.1.5 of Henle.

Note. Recall that the equivalence classes under an equivalence relation, $\sim$, partition the set on which $\sim$ is defined. This is shown in Mathematical Reasoning (MATH 3000); see my online notes for that class on Section 2.9. Set Decomposition: Partitions and Relations and notice Theorem 2.59. Cantor uses the equivalence classes to define the real numbers as follows.

Definition. Let S be the set of all Cauchy sequences of rational numbers. The set of *real numbers*, $\mathbb{R}$, is the set of equivalence classes, $S/\sim$. This version of $\mathbb{R}$ is the *Cantor reals*.

Note RU.B. Before we address the properties of $\mathbb{R} = S/\sim$ (including showing that, with this definition, $\mathbb{R}$ is a complete ordered field), we take a look at the axiomatic development that leads us to the rational numbers. The natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$, are very intuitive to us all. An axiomatic development of $\mathbb{N}$ (along with 0) is given in a senior/graduate level set theory class. Unfortunately, ETSU does not have such a class, but I have (partial) online notes on Introduction to Set Theory; in particular, see Chapter 3. Natural Numbers. Algebraically, $\mathbb{N}$ forms an "additive semigroup." By including the additive inverses of each element of $\mathbb{N}$ (along with 0), we get the integers, $\mathbb{Z}$. The integers form an "abelian additive group." By introducing multiplication on the integers, we then have a "ring." These algebraic structures (namely groups and rings, but also "fields") are covered in Introduction to Modern Algebra (MATH 4127/5127; see my online notes for that class on Section I.4. Groups and Section IV.18. Rings and Fields). To transition from the ring of integers $\mathbb{Z}$ to the field of rational numbers $\mathbb{Q}$, we can do so algebraically by finding the field of quotients of the integers. This is also covered in Introduction to Modern Algebra (MATH 4127/5127) in Section IV.21. The Field of Quotients of an Integral Domain. I have a presentation in PowerPoint posted online on "Integral Domains and Fields of Quotients". This talk has been given in meetings of the ETSU Abstract Algebra Club and the topic was on the club t-shirts in fall 2021 (see below). In connection with the definition of $\mathbb{R}$, we see that the rational numbers $\mathbb{Q}$ are embedded in $\mathbb{R}$. For rational p/q, the equivalence class containing the constant sequence $\{p/q, p/q, \ldots\}$ "represents" p/q.



Note. We now return to showing that Cantor's definition of $\mathbb{R}$ yields a complete ordered field. Since he has defined a real number as an equivalence class of sequences of rational numbers, we have to first define the two binary operations of addition and multiplication. Following Henle's notation, we denote the equivalence class containing Cauchy sequence $\{x(n)\}$ with the bold-faced font $\mathbf{x} \in \mathbb{R}$.

Definition. (Addition and Multiplication of Real Numbers) Let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}$. Let $\{x(n)\}$ be a sequence belonging to equivalence class $\mathbf{x}$, and let $\{y(n)\}$ be a sequence belonging to equivalence class $\mathbf{y}$. Define addition on $\mathbb{R}$ as $\mathbf{x}+\mathbf{y}$ is the equivalence class containing the sequence $\{x(n)+y(n)\}$, and define multiplication on $\mathbb{R}$ as $\mathbf{xy}$ is the equivalence class containing the sequence $\{x(n)y(n)\}$.

Note RU.C. In the previous definition, we have defined addition and multiplication using "representatives" of equivalence classes. That is, $\mathbf{x}$ is represented by $\{x(n)\}$ and $\mathbf{y}$ is represented by $\{y(n)\}$. We are concerned that using different representations of equivalence classes $\mathbf{x}$ and $\mathbf{y}$ might give different values for $\mathbf{x} + \mathbf{y}$. That is, we are concerned about addition and multiplication being *well-defined* (i.e., independent of the choice of representatives). In the next theorem we establish that these two binary operations are, in fact, well-defined.

Theorem 2.1.3. If $\{x(n)\}$ and $\{x'(n)\}$ are equivalent Cauchy sequences of rational numbers, and likewise for $\{y(n)\}$ and $\{y'(n)\}$, then $\{x(n)+y(n)\}$ and $\{x'(n)+y'(n)\}$ are equivalent, and $\{x(n)y(n)\}$ and $\{x'(n)y'(n)\}$ are equivalent.

Note. Henle informally summarizes Theorem 2.1.3 as saying (see his page 40): "equivalent sequences added to equivalent sequences are equivalent, and equivalent sequences multiplied by equivalent sequences are equivalent." More formally, of course, it says that addition and multiplication on $\mathbb{R}$ as defined above are well-defined. We are now equipped to show the algebraic properties of $\mathbb{R}$, namely that $\mathbb{R}$ is a field.

Theorem 2.1.4. $\mathbb{R}$ is a field.

Note RU.D. In the proof of Theorem 2.1.4, we introduced some new notation. We denote the additive identity by $\mathbf{0}$ (the is the equivalence class of null sequences of rational numbers; we will call this "zero"), and denote the multiplicative identity by $\mathbf{1}$. For any real number $\mathbf{x}$, the additive inverse is denoted $-\mathbf{x}$. For any nonzero $\mathbf{x}$ we denote the multiplicative inverse as $\mathbf{x}^{-1}$. We denote the equivalence class of all Cauchy sequence of rational numbers which converge to rational p/q and $\mathbf{p}/\mathbf{q}$; notice that the sequence $(p/q, p/q, \ldots)$ is in $\mathbf{p}/\mathbf{q}$.

Note RU.E. Next, we need to show that $\mathbb{R}$ is ordered. That is, we need to find a positive subset P with the properties given in Axiom 8/Definition of Ordered Field of Section 1.2. The Real Numbers, Ordered Fields. This requires us to define the property of "positive" for a real number; that is, to define a positive equivalence class of Cauchy sequences of rational numbers. This is accomplished as follows, and the definition is shown to be well-defined in the theorem that follows it.

Definition. (Positive Real Number) A Cauchy sequence of rational numbers $\{x(n)\}$ is *positive* if there exist natural numbers M and N so that for $n > N$ we have $x(n) > 1/M$. If $\mathbf{x} \in \mathbb{R}$, then $\mathbf{x}$ is *positive* if one of the sequences in equivalence class $\mathbf{x}$ is positive.

Note RU.F. In Henle's Exercise 2.1.7, the following is to be proved:

> If $\mathbf{x}$ is not zero, and $\{x(n)\}$ is a sequence from $\mathbf{x}$, then there are natural numbers M and N so that when $n > N$, then $|x(n)| > 1/M$.

Since $\mathbf{x}$ is a (nonzero) equivalence class of Cauchy sequences of rational numbers, we see that there are natural numbers M and N such that for all $n > N$ we have *either* $x(n) > 1/M$, or $x(n) < -1/M$ (but not both). In this way, we can partition the real numbers into those that are "positive," those that are "negative," and "zero." This will be done more formally below in Theorem 2.1.6 when we verify the Law of Trichotomy (see "Axiom 8/Definition of Ordered Field" in Section 1.2. Properties of the Real Numbers as an Ordered Field).

Theorem 2.1.5. Let $\mathbf{x} \in \mathbb{R}$. If one sequence from equivalence class $\mathbf{x}$ is positive, then all sequences in $\mathbf{x}$ are positive.

Note. We now have the definitions to show that $\mathbb{R}$ is an ordered field.

Theorem 2.1.6. The field of real numbers $\mathbb{R}$ is ordered.

Note RU.G. Since the Law of Trichotomy holds in this definition of the real numbers, we introduce some terminology for certain real numbers. Real numbers in the set $\{\mathbf{x} \in \mathbb{R} \mid \mathbf{x} \text{ is positive}\}$ are "positive" numbers, real numbers in the set $\{\mathbf{x} \in \mathbb{R} \mid -\mathbf{x} \text{ is positive}\}$ are "negative" numbers, and the real number $\{\mathbf{x} \in \mathbb{R} \mid \mathbf{x} \text{ is null}\} = \mathbf{0}$ is "zero" (as in Note RU.D). As in Section 1.2. Properties of the Real Numbers as an Ordered Field, we say "$\mathbf{a} < \mathbf{b}$" if $\mathbf{b} - \mathbf{a}$ is positive.

Note RU.H. Now we turn to completeness. In Analysis 1 (MATH 4217/5217) we approached the topic of completeness using least upper bounds of bounded sets; see Section 1.3. The Completeness Axiom. Henle calls this *order complete* and when Henle uses the term "complete" in an ordered field he means "order complete."

Completeness Axiom. An ordered field S is (*order*) *complete* if every non-empty subset of S with an upper bound has a least upper bound in S.

When addressing Cauchy sequences, it is easy to show that a convergent sequence is Cauchy (it only requires the Triangle Inequality). But showing that a Cauchy sequence is convergent requires completeness. In Section 2.3. Bolzano-Weierstrass Theorem we used the least upper bound definition of completeness to show that a Cauchy sequence converges (see Exercises 2.3.13 and 2.3.14). An ordered in which Cauchy sequences converge is said to be "Cauchy complete." Formally, we have the following definition.

Definition. (Cauchy Complete Ordered Field) An ordered field S is *Cauchy complete* if every Cauchy sequence in S has a limit in S.

As we will see, we do **not** have an equivalence between order completeness and Cauchy completeness. However, the proofs of Kirkwood's Exercises 2.3.13 and 2.3.14 mentioned above allow us to conclude:

Theorem 1.4.2. If an ordered field is order complete then it is Cauchy complete.

We'll see below that an ordered field that is Cauchy complete *and* in which the Archimedean Principle holds (see Theorem 1-18 in Section 1.3. The Completeness Axiom), is also order complete; see Theorem 1.4.3.

Note RU.I. In many mathematical structures, we can use the definition of Cauchy completeness to *define* a structure as complete if Cauchy sequences converge. This is necessary in settings where there is no ordering (and so there is no concept of an "upper bound" or a "*least* upper bound"). This is the case in the complex setting, for example, where there is no ordering (as is shown in my online notes for Complex Analysis 1 [MATH 5510] on Supplement. Ordering the Complex Numbers). A general setting where completeness is addressed using the convergence of Cauchy sequences is metric spaces. This is illustrated in Introduction to Topology (MATH 4357/5357) (see my online notes for that class on Section 43. Complete Metric Spaces) and in Real Analysis 2 (MATH 5220) (see my online notes for that class on Section 9.4. Complete Metric Spaces; this approach is also covered in Real Analysis 1 [MATH 5210] in the setting of L^p spaces in Section 7.3. L^p is Complete: The Riesz-Fischer Theorem). In *these* notes we clearly distinguish between order complete (which Henle simply calls "complete") and Cauchy complete. In other settings where there is no ordering, there is no such thing as order complete so that in those settings "complete" means Cauchy complete. We now turn to the idea of an Archimedean ordered field.

Definition. (Archimedean Ordered Field) An ordered field S is *Archimedean* if, given two positive numbers a and b, there is $n \in \mathbb{N}$ such that $b < na$.

Note RU.J. In the definition of $\mathbb{R}$ in this supplement, we take $\mathbf{i} \in \mathbb{R}$ to be an *integer* if one of the Cauchy sequences of rational numbers in equivalence class $\mathbf{i}$ is $\{i, i, \ldots\}$ where $i \in \mathbb{Z}$ (and similarly for *natural number* $\mathbf{n}$). Similarly, we take $\mathbf{q} \in \mathbb{R}$ to be *rational* if one of the Cauchy sequences of rational numbers in equivalence class $\mathbf{q}$ is $\{q, q, \ldots\}$. Our approach in this supplement is to consider the "rational numbers" as a fully developed structure with which we are familiar. With this in mind, we illustrate the Archimedean property by proving that the ordered field $\mathbb{Q}$ is Archimedean.

Lemma 1.4.A. (**Exercise 1.4.14** in Henle.) The ordered field of rational numbers $\mathbb{Q}$ is Archimedean.

Note. We now establish that an ordered field which is Cauchy complete and Archimedean, is also order complete. In this we we have the equivalence of these two conditions.

Theorem 1.4.3. An ordered field is order complete if and only if it is Cauchy complete and Archimedean.

Theorem 2.1.A. (**Exercise 2.1.12** in Henle.) The ordered field of real numbers $\mathbb{R}$ is Archimedean.

Note. To address (Cauchy) completeness of the real numbers (as we have defined them), we need to define the absolute value function on $\mathbb{R}$. The definition is not surprising.

Definition. (Absolute Value in Ordered Field) Let S be an ordered field and let $a, b \in S$. The *absolute value* of a is

$$|a| = \begin{cases} a & \text{if } a \text{ is positive or zero,} \\ -a & \text{otherwise.} \end{cases}$$

The *distance* between a and b is $|a - b|$. In particular, with $S = \mathbb{R}$, for $\mathbf{a} \in \mathbb{R}$ we have

$$|\mathbf{a}| = \begin{cases} \mathbf{a} & \text{if } \mathbf{a} \text{ is positive or } \mathbf{a} = \mathbf{0} \\ -\mathbf{a} & \text{if } \mathbf{a} \text{ is negative.} \end{cases}$$

Note RU.K. Since we define real number $\mathbf{x}$ as positive if one of the sequences in equivalence class $\mathbf{x}$ is positive, then we can address absolute value in terms of representatives of the equivalence class (see Theorem 2.1.5 above). Similarly, we can address inequalities in terms of representatives (since "$<$" and "$>$" are defined in terms of the positive set; see Note RU.G). To address completeness we turn our attention to Cauchy sequences of real numbers, so we recall the definition of them. A sequence of real numbers $\{\mathbf{x}(n)\}$ is a Cauchy sequence if for any given positive real number $\boldsymbol{\varepsilon} > \mathbf{0}$ there is a natural number $N(\boldsymbol{\varepsilon})$ such that for all $m, n > N(\boldsymbol{\varepsilon})$ we have $|\mathbf{x}(n) - \mathbf{x}(m)| < \boldsymbol{\varepsilon}$ (that is, $\boldsymbol{\varepsilon} - |\mathbf{x}(n) - \mathbf{x}(m)| > \mathbf{0}$). Since we deal with the condition "$\boldsymbol{\varepsilon} - |\mathbf{x}(n) - \mathbf{x}(m)| > \mathbf{0}$" in terms of representatives of the equivalence classes, we will consider $\{x(n,i)\}_{i=1}^{\infty} \in \mathbf{x}(n)$, $\{x(m,i)\}_{i=1}^{\infty} \in \mathbf{x}(m)$ and $\{e(i)\}_{i=1}^{\infty} \in \boldsymbol{\varepsilon}$. With $\mathbf{x}(n)$ as a Cauchy sequence, then for all $m, n > N(\boldsymbol{\varepsilon})$ the sequence of rational numbers $\{e(i) - |x(n,i) - x(m,i)|\}_{i=1}^{\infty}$ is positive (and conversely). Notice that the definition of positive sequence gives us $\{e(i) - |x(n,i) - x(m,i)|\}_{i=1}^{\infty} > 0$ if and only if there are natural numbers M and N such that for $i > N$ we have

$e(i) - |x(n,i) - x(m,i)| > 1/M$. So the condition of sequence of real numbers $\{\mathbf{x}(n)\}$ being Cauchy is equivalent to: for all real $\boldsymbol{\varepsilon} > \mathbf{0}$, there exists natural number $N(\boldsymbol{\varepsilon})$ such that for all $m, n > N(\boldsymbol{\varepsilon})$ we have for some natural numbers M and N, if $i > N$ then $e(i) - |x(n,i) - x(m,i)| > 1/M$.

Note RU.L. Similar to the previous note, the definition of convergence of sequence of real numbers $\{\mathbf{x}(n)\}$ to real number $\mathbf{b}$ in the ordered field $\mathbb{R}$ states that for all positive $\boldsymbol{\varepsilon} > \mathbf{0}$, there is natural number $N(\boldsymbol{\varepsilon})$ such that for all $n > N(\boldsymbol{\varepsilon})$ we have $|\mathbf{x}(n) - \mathbf{b}| < \boldsymbol{\varepsilon}$. Since we deal with the condition "$\boldsymbol{\varepsilon} - |\mathbf{x}(n) - \mathbf{b}| > \mathbf{0}$" in terms of representatives of the equivalence classes, we will consider $\{x(n,i)\}_{i=1}^{\infty} \in \mathbf{x}(n)$, $\{b(i)\}_{i=1}^{\infty} \in \mathbf{b}$, and $\{e(i)\}_{i=1}^{\infty} \in \boldsymbol{\varepsilon}$. With $\mathbf{x}(n)$ convergent to $\mathbf{b}$, for all $n > N(\boldsymbol{\varepsilon})$ the sequence of rational numbers $\{e(i) - |x(n,i) - b_i|\}_{i=1}^{\infty}$ is positive (and conversely). Notice that the definition of positive sequence gives us $e(i) - |x(n,i) - b_i| > 0$ if and only if there are natural numbers M and N such that for $i > N$ we have $e(i) - |x(n,i) - b_i| > 1/M$. So the condition of sequence of real numbers $\{\mathbf{x}(n)\}$ being convergent to real number $\mathbf{b}$ is equivalent to: for all real $\boldsymbol{\varepsilon} > \mathbf{0}$, there exists natural number $N(\boldsymbol{\varepsilon})$ such that for all $n > N(\boldsymbol{\varepsilon})$ we have for some natural numbers M and N, if $i > N$ then $e(i) - |x(n,i) - b_i| > 1/M$.

Note. We now show that the real numbers, as defined in terms of equivalence classes of Cauchy sequences of rational numbers, are a (Cauchy) complete ordered field. Below, we will define isomorphisms of complete ordered fields and prove that, up to isomorphism, there is only one complete ordered field.

Theorem 2.1.7. The real numbers $\mathbb{R}$ form an order complete ordered field.

Note RU.M. We now show that all complete ordered fields are isomorphic. So there is "only one" complete ordered field ("up to isomorphism"), namely the real numbers $\mathbb{R}$. This means that the axioms of the real numbers are *categorical*. That is, each model of the real numbers is isomorphic to every other model. The topic of a categorical axiomatic system is covered in Introduction to Modern Geometry (MATH 4157/5157); see my online notes for that class on Section 1.6. Completeness and Categoricalness. So far in this supplement, we have taken the real numbers to be equivalence classes of Cauchy sequences of rational numbers (working under the assumption that the rational numbers $\mathbb{Q}$ have already been developed, as described in Note RU.B). Later we will define the real numbers as "Dedekind cuts." Henle speculates on his pages 46 and 47 that one might as: "What are the reals *really*? Are they equivalence classes of Cauchy sequences? Are they Dedekind cuts? Or what? ... For some the nature of the real numbers is not settled by these constructions; it remains a problem in the philosophy of mathematics." Since all models of the real numbers are (effectively) the same, the *mathematics* of the real numbers is not affected by the answers to these questions; only the *philosophy* is affected or influenced by the answers. We now address this uniqueness of $\mathbb{R}$. This material is based on Henle's Section 2.3, "Uniqueness of the Reals."

Definition. Let S be an ordered field with multiplicative identity denoted "1_S," and additive identity denoted "0_S." Define $i : \mathbb{Z} \to S$ as

$$i(n) = \begin{cases} 1_S + 1_S + \cdots + 1_S & n \text{ times, for } n > 0 \\ 0_S & \text{for } n = 0 \\ -1_S - 1_S - \cdots - 1_S & -n \text{ times, for } n < 0. \end{cases}$$

Elements of S of the form $i(n)$ where $n \in \mathbb{Z}$ are called *integers* themselves. Extend i to $\mathbb{Q}$ by defining, for $p/q \in \mathbb{Q}$, $i(p/q) = i(p)i(q)^{-1} = i(p)/i(q)$. We use the following notation to denote certain subsets of S: $S_{\mathbb{N}} = \{i(n) \mid n \in \mathbb{N}\}$, $S_{\mathbb{Z}} = \{i(n) \mid n \in \mathbb{Z}\}$, $S_{\mathbb{Q}} = \{i(p/q) \mid p/q \in \mathbb{Q}\}$. These sets are called the *natural numbers*, *integers*, and *rational numbers* in ordered field S.

Note RU.N. We know by Kirkwood's Exercise 1.2.7(a) that, in ordered field S, $1_S > 0_S$; that is, 1_S is positive. Since the positive set is closed under addition, then we see that for $n \in \mathbb{N}$ we have $i(n)$ is positive in S. Similarly, $-1_S < 0$ so that for negative integer n we have that $i(n)$ is negative in S. In a field F where there exists $n \in \mathbb{N}$ such that for all $a \in F$ we have $\underbrace{a + a + \cdots + a}_{n \text{ times}} = 0$, the field is said to have *characteristic* n. A field can have characteristic p where p is prime, for example the integers modulo p form the field $\mathbb{Z}_p$. However, we see that there can be no such characteristic n for an ordered field (a field that is not characteristics n for any $n \in \mathbb{N}$ is said to have characteristic 0). The idea of "characteristic" is defined in the setting of rings in Introduction to Modern Algebra 1 (MATH 4127/5127); see my online notes for that class on Section IV.19. Integral Domains and notice Definition IV.19.13.

Note. Since a given rational number can be expressed in more than one way as a quotient, for example $1/2 = 2/4 = 3/6$, then we need to confirm that $i : \mathbb{Q} \to S$ is well-defined. This is accomplished in the following three results.

Theorem 1.3.1. The function $i : \mathbb{N} \cup \{0\} \to S$, where S is an ordered field, satisfies:

(a) $i(n + m) = i(n) + i(m)$ for all $m, n \in \mathbb{N} \cup \{0\}$,

(b) $i(nm) = i(n)i(m)$ for all $m, n \in \mathbb{N} \cup \{0\}$, and

(c) i is one to one on $\mathbb{N} \cup \{0\}$.

Note. Theorem 1.3.1 implies that $S_{\mathbb{N}}$ is a "copy" of the natural numbers in an ordered field S. For this reason, Theorem 1.3.1 is called an "embedding theorem." The next result concerns the embedding of integers in S. It's proof is to be given in Henle's Problem 1.3.15.

Theorem 1.3.2. The function $i : \mathbb{Z} \to S$, where S is an ordered field, satisfies:

(a) $i(n + m) = i(n) + i(m)$ for all $m, n \in \mathbb{Z}$,

(b) $i(nm) = i(n)i(m)$ for all $m, n \in \mathbb{Z}$, and

(c) i is one to one on $\mathbb{Z}$.

Note. The mapping $i : \mathbb{Z} \to S$ is an example of a *ring isomorphism* and the mapping $i : \mathbb{Q} \to S$ is an example of a *field isomorphism* (as we'll show below in Theorem 2.3.B). These topics are covered in Introduction to Modern Algebra 1 (MATH 4127/5127); see my online notes for that class on Section IV.18. Rings and Fields. As above, we say that "$\mathbb{Z}$ is embedded in S" (meaning S contains an isomorphic of $\mathbb{Z}$, namely $S_{\mathbb{Z}}$), and "$\mathbb{Q}$ is embedded in S" (meaning S contains an isomorphic of $\mathbb{Q}$, namely $S_{\mathbb{Q}}$; we prove that i is an ordered field isomorphism in Theorem 2.3.1 below).

Theorem 2.3.B. (Problem 2.3.1) Mapping $i : \mathbb{Q} \to S$ is well-defined. That is, $i(p/q) = i(r/s)$ for $p/q = r/s$ where $p, q, r, s \in \mathbb{Z}$.

Note. We now have the equipment to show that i is an ordered field isomorphism between $\mathbb{Q}$ and $S_{\mathbb{Q}}$.

Theorem 2.3.1. The function $i : \mathbb{Q} \to S$ is a field and order isomorphism from the $\mathbb{Q}$ onto a subfield of S.

Note. We now show that the rational numbers in S are distributed similarly to the rational numbers in $\mathbb{R}$. That is, we show that $S_{\mathbb{Q}}$ is dense in S. The next theorem is analogous to Kirkwood's Exercise 1.3.4(a) (see the notes on Section 1.3. The Completeness Axiom for a solution to that exercise). It's proof is the same as that of Kirkwood's exercise (since it only uses properties of the real numbers as a complete ordered field).

Theorem 2.3.2. Let a and b be any two elements of complete ordered field S and assume that $a < b$. Then there is a rational number q between a and b; that is, there are integers m and n in S such that if $q = m/n$ then $a < q < b$.

Note. We can now prove the main result of this supplement.

Theorem 2.3.3. Every order complete ordered field is isomorphic to $\mathbb{R}$ (where we take $\mathbb{R}$ to be the complete ordered field of equivalence classes of Cauchy sequences of rational numbers).

Note. The model of $\mathbb{R}$ as equivalence classes of Cauchy sequences of rational numbers is called the *Cantor reals*. We conclude this supplement with another model of $\mathbb{R}$ called the *Dedekind reals*. We give less details of this model, though we will claim that the Dedekind reals form a complete ordered field. We know by Theorem 2.3.3 that the Cantor reals and the Dedekind reals are isomorphic. As discussed in Note RU.A, Dedekind was the first to give a construction of the real numbers in the 1850s, though he did not publish his results until 1872 (coincidentally, the same year that Cantor published his work on the real numbers). As Cantor did, Dedekind also starts with the rationals $\mathbb{Q}$.

Definition. A *Dedekind cut* (or just *cut*) is a subset $\mathbf{x}$ of the rational $\mathbb{Q}$ such that

(a) neither $\mathbf{x}$ not the complement of $\mathbf{x}$ is empty,

(b) if r is in $\mathbf{x}$, and $s > r$, then s is in $\mathbf{x}$, and

(c) $\mathbf{x}$ has no least element.

The set of *real numbers*, $\mathbb{R}$, is the set of all Dedekind cuts. This version of $\mathbb{R}$ is also called the *Dedekind reals.*

Note. In Henle's Problem 2.2.2 it is to be shown that for $x \in \mathbb{Q}$, $\{r \in \mathbb{Q} \mid r > x\}$ is a Dedekind cut, called a *rational cut.* The special rational cut $\{r \in \mathbb{Q} \mid r > 0\}$ is the *null* cut. Notice that for $x \in \mathbb{Q}$, he set $\{r \in \mathbb{Q} \mid r \geq x\}$ is not a cut, because it has a least element (namely r) and so violates part (c) of the definition of Dedekind cut. For x an irrational number, informally think of a Dedekind cut as $\mathbf{x} = \{r \in \mathbb{Q} \mid r > x\}$; of course we cannot use an irrational number x to define an "irrational cut," but this is where the results of this section will ultimately (and rigorously) lead. We need to define addition, multiplication, and positive on $\mathbb{R}$ (as defined by Dedekind).

Definition. If $\mathbf{x}$ and $\mathbf{y}$ are cuts, then the *sum* $\mathbf{x} + \mathbf{y}$ is

$$\mathbf{x} + \mathbf{y} = \{r + s \mid r \in \mathbf{x} \text{ and } s \in \mathbf{y}\}.$$

Theorem 2.2.1/2.2.2. Addition of cuts is well-defined (i.e., $\mathbf{x} + \mathbf{y}$ is a cut). Addition on $\mathbb{R}$ satisfies the laws of commutativity, associativity, additive identity, and additive inverse.

Note. The proof of Theorem 2.2.1 is to be given in Henle's Problem 2.2.5. The proof of Theorem 2.2.2 is to be given in Problem 2.2.6.

Definition. A cut is *non-negative* if it is a subset of the null cut. It is *positive* if it is a proper subset of the null cut. The *negative* of cut $\mathbf{x}$ is

$$-\mathbf{x} = \{-r | r \text{ is neither in } \mathbf{x} \text{ nor is the glb of } \mathbf{x}\}.$$

Theorem 2.2.3. The positive cuts are closed under addition. The Dedekind reals satisfy the Law of Trichotomy.

Theorem 2.2.4. $\mathbb{R}$ is order complete.

Note. The proof of Theorem 2.2.3 is to be given in Henle's Problem 2.2.7. The proof of Theorem 2.2.4 is to be given in Problem 2.2.9. In Problem 2.2.8, it to be shown that $\mathbf{x} < \mathbf{y}$ if and only if $\mathbf{x} \supseteq \mathbf{y}$. We now turn to multiplication.

Definition. If $\mathbf{x}$ and $\mathbf{y}$ are non-negative cuts, the *product* of $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$\mathbf{xy} = \{rs \mid r \in \mathbf{x} \text{ and } s \in \mathbf{y}\}.$$

Theorem 2.2.5. Multiplication is well-defined for non-negative cuts and the set of non-negative cuts is closed under multiplication. In addition, multiplication satisfies the laws of commutativity, associativity, identity, inverse, and the distributive law (for non-negative cuts).

Note. Since we have not addressed products of negative cuts, we cannot yet say that $\mathbb{R}$ is a field. When dealing with the properties mentioned in Theorem 2.2.5 for possibly negative cuts, it is required that cases under consideration must consider the "sign" (positive or negative) of the involved cuts. Since we have an ordering and the Law of Trichotomy holds (by Theorem 2.2.3), then for every nonzero real number $\mathbf{x}$ either $\mathbf{x}$ is non-negative or $-\mathbf{x}$ is non-negative. This allows us to define absolute value as follows.

Definition. If $\mathbf{x}$ is a non-null cut, then exactly one of $\mathbf{x}$ and $-\mathbf{x}$ is non-negative. Define the *absolute value* of $\mathbf{x}$ as

$$|\mathbf{x}| = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \text{ is non-negative} \\ -\mathbf{x} & \text{if } -\mathbf{x} \text{ is non-negative.} \end{cases}$$

If $\mathbf{x}$ is a null cut, define $|\mathbf{x}| = \mathbf{x}$. For cuts $\mathbf{x}$ and $\mathbf{y}$ define the *product* of $\mathbf{x}$ and $\mathbf{y}$ as

$$\mathbf{xy} = \begin{cases} |\mathbf{x}|\,|\mathbf{y}| & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ are both positive,} \\ |\mathbf{x}|\,|\mathbf{y}| & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ are both negative,} \\ -|\mathbf{x}|\,\mathbf{y} & \text{otherwise.} \end{cases}$$

Theorem 2.2.6. $\mathbb{R}$ is an order complete ordered field.

Note. Henle declares (see his page 46): "The proof of this theorem is tedious since the verification of each axiom breaks down into many separate cases depending on the signs of the different quantities." A proof can be found in Claude Burrill's

Foundations of Real Numbers, McGraw-Hill (1967); see pages 126 to 130. As observed above, Theorem 2.3.3 implies that the Cauchy reals and the Dedekind reals are isomorphic (order) complete ordered fields.

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