Analysis 1, Chapter 2 Study Guide

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The following is a *brief* list of topics covered in Chapter 2 of Kirkwood's *An Introduction to Analysis*, and the associated supplements. This list is not meant to be comprehensive, but gives a list of several important topics.

Chapter 2. Sequences of Real Numbers.

2-1. Sequences of Real Numbers. Sequence, convergent sequence, limit of a sequence, divergent sequence, the idea behind the definition of limit of a sequence (Note 2.1.A), proofs using the definition of limit of a sequence (Examples 2.1.A and 2.4), diverge to infinity, diverge to negative infinity, proof that a sequence diverges to infinity (Example 2.6), limits of a sequence are unique (Theorem 2-1), ε condition for a convergent sequence (Theorem 2-2), bounded sequence, convergent sequences are bounded (Theorem 2-3), sums/products/quotients/multiples of sequences, limits of sums/products/quotients/multiples of sequences (Theorem 2-4), limits of sequences and inequalities (Theorem 2-5), monotone increasing/decreasing sequence, a bounded monotone sequence converges (Theorem 2-6), limits of monotone increasing/decreasing sequences (Corollary 2-6), proof that a sequence is monotone (Example 2.9), nested sequence, a nested sequence of closed intervals whose length has limit 0 has a single point in all intervals (Theorem 2.7), the lub of a set is the limit of a sequence of elements of the set (Theorem 2-8), Cauchy sequence, theoretical importance of Cauchy sequences and their use in defining completeness (Note 2.1.C), a sequence converges if and only if it is Cauchy (Theorem 2-9), history of continuum and completeness and the roles played by Bolzano and Cauchy (Note 2.1.D).

2-2. Subsequences. Subsequence, double indices in a subsequence (Note 2.2.A), subsequential limit, ∞ and $-\infty$ are considered subsequential limits (Note 2.2.B), convergent sequences and convergent subsequences (Theorem 2-10), ε classification of subsequential limits (Theorem 2-11 and Corollary 2-11), monotone subsequences of convergent sequences (Exercise 2.2.12(a)), sequences with lots of subsequential limits (Note 2.2.C).

2-3. The Bolzano-Weierstrass Theorem. Limit point of a set of real numbers, every bounded infinite set of real numbers has a limit point (Bolzano-Weierstrass Theorem; Theorem 2-12), brief biography of Karl Weierstrass (Note 2.3.A), classification of subsequential limits in terms of limit limit points of the set of terms of the sequence (Theorem 2-13), a bounded sequence has a convergent subsequence (Theorem 2-14), unbounded sequences and subsequences that diverge to $\pm\infty$ (Theorem 2-15), classification of convergent sequences in terms of boundedness and subsequential limits (Theorem 2-16), $\limsup a_n = \overline{\lim}\, a_n$, $\liminf a_n = \underline{\lim}\, a_n$, alternate definition of $\overline{\lim}$ and $\underline{\lim}$ (Note 2.3.B), $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$ are subsequential limits of $\{a_n\}$ (Exercise 2.3.16), ε classification of $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$ (Theorem 2-17), convergence of a bounded sequence in terms of $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$ (Corollary 2-17), $\overline{\lim}$ and $\underline{\lim}$ of sums of sequences (Theorem 2-18), bounded function, $\sup(f)$ and $\inf(f)$ for function f, sup and inf of a sum of functions (Theorem 2-19), properties of Cauchy sequences and proof that a Cauchy sequence converges (Exercise 2.3.13), convergent sequences are Cauchy (Exercise 2.3.14).

Supplement. The Real Numbers are the Unique Complete Ordered Field. Brief biographies of Richard Dedekind and Georg Cantor (Note RU.A), null sequence, sums and products of Cauchy sequences and null sequences (Theorem 2.1.1), the equivalence relation $\sim$ on sequences of rational numbers and the equivalence classes of $\sim$, $\sim$ is an equivalence relation (Theorem 2.1.2), definition of $\mathbb{R}$ in terms of equivalence classes of sequences of rational numbers, axiomatic development of $\mathbb{N}$ and $\mathbb{Z}$ and $\mathbb{Q}$ (Note RU.B), definition of addition of equivalence classes, well-defined operations (Note RU.C), proof that addition and multiplication of equivalence classes are well-defined (Theorem 2.1.3), $\mathbb{R}$ is a field, ordering $\mathbb{R}$ (Note RU.D), definition of a positive Cauchy sequence of rational numbers, proof that the property of positive on equivalence classes is well-defined (Theorem 2.1.5), $\mathbb{R}$ is an ordered field, order complete and the fact that the convergence of Cauchy sequences is equivalent to our Axiom of Completeness (Note RU.E), the use of Cauchy sequences to define completeness in settings where there is no ordering (Note RU.F), Cauchy complete ordered field, Archimedean ordered field, $\mathbb{R}$ is Archimedean (Theorem 2.1.A/Exercise 2.1.12), $\mathbb{R}$ is a complete ordered field (Theorem 2.1.7).