Supplement. A Classification of Open Sets of Real Numbers

Note. In this supplement, we prove that a set of real numbers is open if and only if it is a countable union of disjoint open intervals. **This is the most fundamental result of the Analysis 1 and 2 sequence!!!** It will be mentioned repeatedly in graduate-level Real Analysis 1 (MATH 5210). It also plays a role in Introduction to Topology (MATH 4357/5357).

Note. Our goal in this supplement is to give a proof of the following:

Theorem 3-5. A set of real numbers is open if and only if it is a countable union of disjoint intervals.

Note. We need two preliminary results which are Exercises in James R. Kirkwood's *An Introduction to Analysis* 2nd Edition (Waveland Press Incorporated, 1995).

Exercise 3.1.4. Let A be an open set containing x. Let $\alpha = \sup\{\delta \mid [x, x+\delta) \subset A\}$ and $\beta = \sup\{\delta \mid (x-\delta, x] \subset A\}$. Assume α and β are finite. Then $(x-\beta, x+\alpha) \subset A$, $x + \alpha \notin A$, and $x - \beta \notin A$.

Note OS.A. For A an open set containing x, define the interval $I_x = (x - \alpha, x + \beta)$ where α and β are as defined in Exercise 3.1.4. If no such finite α or β exists, then we take $I_x = (-\infty, x + \beta)$, $I_x = (x - \alpha, \infty)$, or $I_x = (-\infty, \infty)$, as needed. We only have $I_x = (-\infty, \infty)$ with $A = \mathbb{R}$. Notice that any open interval containing x that is a subset of A, must be a subset of I_x; otherwise I_x would have to include either $x + \alpha$ or $x - \beta$, but this cannot be the case by Exercise 3.1.4. Such an interval I_x is called the *maximal open interval* of A which contains x.

Exercise 3.1.5. Let A be an open set. Let I_x be the maximal open interval containing x. Then either $I_x = I_y$ or $I_x \cap I_y = \varnothing$.

Note. A *partition* of a set is a collection of disjoint subsets that union to give the set. See my online notes for Mathematical Reasoning (MATH 3000) on Section 2.9. Set Decomposition: Partitions and Relations; notice Definition 2.47. Therefore, by Exercise 3.1.5, we see that an open set A of real numbers is partitioned by the set of distinct I_x, $\{I_x \mid x \in A\}$.

Note. We now have the background to classify open sets of real numbers. The classification is in terms of open intervals. The open intervals are the maximal open subintervals of Note OS.A. We'll also see that there are only countably many such (distinct) open subintervals.

Theorem 3-5. Classification of open sets of real numbers.

A set of real numbers is open if and only if it is a countable union of disjoint open intervals.

Note OS.B. Theorem 3-5 is a "classification theorem." It gives necessary and sufficient conditions for a set of real numbers to be open. It tells us what an open set of real numbers *looks like*. In graduate level Real Analysis 1 (MATH 5210), attempts will be made to describe what other sets of real numbers look like. None of those other attempts will be as successful as our description of open sets! As we'll see below, out description of closed sets lacks the elegance of Theorem 3-5. We know by Theorem 3-2 that an arbitrary union of open sets is open, but only a finite intersection of open sets is necessarily open. So what is we considered bigger intersections of open sets? One class of sets that will be discussed in Real Analysis 1 are "G_δ" sets of real numbers. A set of real numbers is in the class G_δ if it is a countable intersection of open sets. Notice that a "countable" restriction is imposed on the G_δ sets, similar to the countable property that we have in Theorem 3-5. Since Theorem 3-5 tells us what an open set looks like, we can describe what a G_δ set looks like:

> A set of real numbers is G_δ if it is a countable intersection of a countable union of (disjoint) open intervals.

I told you things would not be as elegant as they are for open sets!

Note. A set of real numbers is, by definition, closed if its complement if open. So we trivially have the following.

Theorem OS.1. Classification of closed sets of real numbers.
A set of real numbers is closed if and only if its complement is a countable union of disjoint open intervals.

Note OS.C. We might hope for a more direct classification of closed sets of real numbers. One might *guess* that a closed set of real numbers is a countable union of disjoint closed intervals. However, this is not the case. Consider the countable collection of closed intervals $[1/n, 1/(n-1/2)]$ for $n \in \mathbb{N}$. Notice that 0 is a limit point of $\cup_{n=1}^{\infty}[1/n, 1/(n-1/2)]$ (since the sequence of elements of the set, $\{1/n\}$, converges to 0). But 0 is not in the set, so the set is not closed by Corollary 3-6(a). We have some classifications of closed sets (see, for example, Theorem 3-6 where closed sets are classified in terms of their boundary points, and Corollary 3-6(a) where closed sets are classified in terms of their limit points), but we don't have a (direct) classification of closed sets which is as elegant as the classification of open sets (as given in Theorem 3-5).

Note OS.D. We know by Theorem 3-3 that an arbitrary intersection of closed sets is closed, but only a finite union of closed sets is necessarily closed. Similar to Note OS.B, we now consider bigger unions of closed sets. Another class of sets that will be discussed in Real Analysis 1 are "F_σ" sets of real numbers. A set of real numbers is in the class F_σ if it is a countable union of closed sets. Similar to the description of what a G_δ set looks like, we can describe what a G_δ set looks like (but as discussed in Note OS.C, we still struggle to describe closed sets):

> A set of real numbers is F_σ if it is a countable union of sets which are complements of countable unions of (disjoint) open intervals.

Elegance is again lacking because all we have for describing what closed sets of real numbers look like is Theorem OS.1.

Note OS.E. The purpose of introducing the classes G_δ and F_σ of sets of real numbers (think of them as classes that go beyond the classes of "open" and "closed" sets) is to consider some sets of real numbers for which we have some kind of description (inelegant though it may be). In the development of measure theory in Real Analysis 1, we'll that open sets, closed sets, G_δ sets, and F_σ sets are examples *Borel sets* (all of which are measurable sets). But there are many other classes of measurable sets. Remember, there are uncountably many real numbers (by Theorem 1-20, say), and even "more" *sets* of real numbers (by Cantor's Theorem, Theorem 1-21). With the introduction of these classes of sets, we are doing what we can! For more information on these classes of sets, see my online Real Analysis 1 notes on Section 1.4. Borel Sets.

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