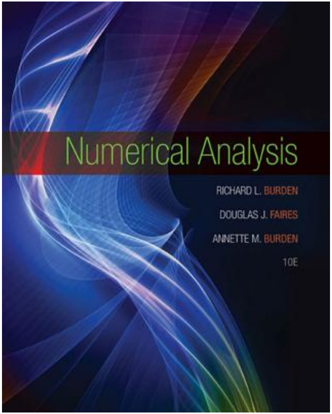
Numerical Analysis

Chapter 2. Solutions of Equations in One Variable

2.2. Fixed-Point Iteration—Proofs of Theorems



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(i) If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has at least on fixed point in $[a, b]$.

(ii) If, in addition, $g'(x)$ exists on (a, b) and a positive constant $k < 1$ exists with $|g'(x)| \leq k$ for all $x \in (a, b)$, then there is exactly on fixed point in $[a, b]$.

Proof. (i) If $g(a) = 1$ or $g(b) = b$, then g has a fixed point at an endpoint of $[a, b]$. If not, then $g(a) > a$ and $g(b) < b$. The function $h(x) = g(x) - x$ is continuous on $[q, b]$, with

$$h(a) = g(a) - a > 0 \text{ and } h(b) = g(b) - b < 0.$$

The Intermediate Value Theorem (see my online notes for Calculus 1 [MATH 1910] on Section 2.5. Continuity; see Theorem 2.11) implies that there is $p \in (a, b)$ for which $h(p) = 0$. The number p is a fixed point for g because $0 = h(p) = g(p) - p$, or $g(p) = p$.

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Theorem 2.4

Theorem 2.4. Fixed-Point Theorem.
Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose, in addition, that g' exists on (a, b) and that a constant $0 < k < 1$ exists with $|g'(x)| \leq k$ for all $x \in (a, b)$. Then for any number $p_0 \in [a, b]$, the sequence defined by $p_n = g(p_{n-1})$, for $n \in \mathbb{N}$, converges to the unique fixed point $p \in [a, b]$.

Proof. By Theorem 2.3(ii) there is unique point p in $[a, b]$ with $g(p) = p$. Since g maps $[a, b]$ into itself, the sequence $\{p_n\}_{n=0}^{\infty}$ is defined for all $n \geq 0$ and $p_n \in [a, b]$ for all n. Since $|g'(x)| \leq k$, then by the Mean Value Theorem we have for each n that

$$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(\xi_n)||p_{n-1} - p| \leq k|p_{n-1} - p|,$$

for some ξ_n between p_{n-1} and p (so that $\xi_n \in (a, b)$). This inequality then implies

$$|p_n - p| \leq k|p_{n-1} - p| \leq k^2|p_{n-2} - p| \leq \cdots \leq k^n|p_0 - p|. \qquad (2.4)$$

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Since $0 < k < 1$ then $\lim_{n\to\infty} k^n = 0$ and so

$$\lim_{n\to\infty} |p_n - p| \leq \lim_{n\to\infty} k^n|p_0 - p| = 0.$$

Hence, $\{p_n\}_{n=0}^{\infty}$ converges (by the definition of limit of a sequence) to fixed point p, as claimed. □

Corollary 2.5

Corollary 2.5. If g satisfies the hypotheses of Theorem 2.4, then bounds for the error involved in using p_n to approximate p are given by $|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$ and $|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|$ for all $n \in \mathbb{N}$.

Proof. Notice for $p_0 \in [a, b]$, $p_0 - a$ is the distance from p_0 to the left-hand endpoint of $[a, b]$ and $b - p_0$ is the distance from p_0 to the right-hand endpoint of $[a, b]$. Hence for $p, p_0 \in [a, b]$ we have $|p_0 - p| \leq \max\{p_0 - a, b - p_0\}$. So from Inequality (2.4) in the proof of Theorem 2.4 we have $|p_n - p| \leq k^n|p_0 - p| \leq k^n \max\{p_0 - a, b - p_0\}$.

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$$|p_{n+1} - p_n| = |g(p_n) - g(p_{n-1})| = |g'(\xi_n)||p_n - p_{n-1}| \leq k|p_n - p_{n-1}|,$$

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Proof (continued). Iterating this process we have for $n \ge 1$ that

$$|p_{n+1} - p_n| \le k|p_n - p_{n-1}| \le k^2|p_{n-1} - p_{n-2}| \le \cdots \le k^n|p_1 - p_0|.$$

Thus for $m > n \ge 1$ we have

$$\begin{aligned} |p_m - p_n| &= |p_m - p_{m-1} + p_{m-1} - \cdots - p_{n+1} - p_n| \\ &\le |p_m - p_{m-1}| + |p_{m-1} - p_{m-2}| + \cdots + |p_{n+1} - p_n| \\ &\le k^{m-1}|p_1 - p_0| + k^{m-2}|p_1 - p_0| + \cdots + k^n|p_1 - p_0| \\ &= k^n|p_1 - p_0|(1 + k + k^2 + \cdots + k^{m-n-1}). \end{aligned}$$

Corollary 2.5 (continued 2)

Corollary 2.5. If g satisfies the hypotheses of Theorem 2.4, then bounds for the error involved in using p_n to approximate p are given by $|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$ and $|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|$ for all $n \in \mathbb{N}$.

Proof (continued). As seen in the proof of Theorem 2.3, the sequence $\{p_n\}_{n=0}^{\infty}$ converges of p, and so

$$|p - p_n| = \lim_{m\to\infty} |P_m - p_n| \leq \lim_{m\to\infty} k^n|p_1 - p_0| \sum_{i=1}^{m-n-1} k^i$$

$$\leq k^n|p_1 - p_0| \sum_{i=0}^{\infty} k^i = \frac{k^n}{1-k}|p_1 - p_0|$$

since $\sum_{i=0}^{\infty} k^i = 1/(1-k)$ because this is the sum of a geometric series with ratio k where $0 < k < 1$. This is the claimed bound on $|p - p_n|$. □