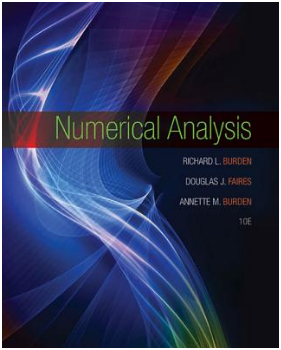
Numerical Analysis

Chapter 3. Interpolation and Polynomial Approximation

3.1. Interpolation and the Lagrange Polynomial—Proofs of Theorems



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Theorem 3.3

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$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n),$$

where $P(x)$ is the nth Lagrange interpolating polynomial.

Proof. Notice first for $x = x_k$ for any $k = 0, 1, \ldots, n$, that $f(x_k) = P(x_k)$ regardless of the value of $\xi(x_k)$. If $x \neq x_k$ for $k = 0, 1, \ldots, n$ then define function g for $t \in [a, b]$ as

$$g(t) = f(t) - P(t) - (f(x) - P(x))\frac{(t - x_0)(t - x_1)\cdots(t - x_n)}{(x - x_0)(x - x_1)\cdots(x - x_n)}$$

$$= f(t) - P(t) - (f(x) - P(x))\prod_{i=0}^{n}\frac{(t - x_i)}{(x - x_i)}.$$

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Theorem 3.3 (continued 1)

Proof (continued). For $t = x_k$ we have

$$\begin{aligned} g(x_k) &= f((x_k)) - P((x_k)) - (f(x) - P(x)) \prod_{i=0}^{n} \frac{((x_k) - x_i)}{(x - x_i)} \\ &= 0 - (f(x) - P(x)) \cdot 0 = 0. \end{aligned}$$

Moreover, with $t = x \neq x_i$ for $i = 0, 1, \ldots, n$ (think of x as otherwise arbitrary, but fixed) then

$$\begin{aligned} g(x) &= f((x)) - P((x)) - (f(x) - P(x)) \prod_{i=0}^{n} \frac{((x) - x_i)}{(x - x_i)} \\ &= f(x) - P(x) - (f(x) - P(x)) \cdot 1 = 0. \end{aligned}$$

That is, $g \in C^{n+1}[a, b]$ and g is zero at the $n + 2$ distinct numbers $x, x_0, x_1, \ldots, x_n$. By the Generalized Rolle's Theorem (Theorem 1.10), there exists a number $\xi(x) = \xi \in (a, b)$ (based on fixed x) for which $g^{(n+1)}(\xi) = 0$.

Theorem 3.3 (continued 2)

Proof (continued). So differentiating $g(t)$ $n+1$ times and then taking $t = \xi$ we have

$$\begin{aligned} 0 &= g^{(n+1)}(\xi) = f^{(n+1)}(\xi) - P^{(n+1)}(\xi) \\ &\quad -(f(x) - P(x))\frac{d^{(n+1)}}{dt^{(n+1)}}\left[\prod_{i=0}^{n}\frac{(t-x_i)}{(x-x_i)}\right]_{t=\xi}. \qquad (*) \end{aligned}$$

But P is a polynomial of degree at most n, so the $(n+1)$th derivative, $P^{(n+1)}$, is the zero function. Also, $\displaystyle\prod_{i=0}^{n}\frac{(t-x_i)}{(x-x_i)}$ is a polynomial in t of degree $(n+1)$, so

$$\prod_{i=0}^{n}\frac{(t-x_i)}{(x-x_i)} = \left(\frac{1}{\prod_{i=0}^{n}(x-x_i)}\right)t^{n+1} + (\text{lower-degree terms in } t), \ldots$$

Theorem 3.3 (continued 3)

Proof (continued). ...and so differentiating $n+1$ times with respect to t gives

$$\frac{d^{n+1}}{dt^{n+1}}\left[\prod_{i=0}^{n}\frac{t-x_i}{x-x_i}\right]=\frac{(n+1)!}{\prod_{i=0}^{n}(x-x_i)}.$$

Now $(*)$ becomes

$$0=f^{(n+1)}(\xi)-0-(f(x)-P(x))\frac{(n+1)!}{\prod_{i=0}^{n}(x-x_i)},$$

and so

$$f(x)=P(x)+\frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x-x_i).$$

Now this has been demonstrated for $x\neq x_i$ where $i=0,1,\ldots,n$, but it also "clearly" holds for these x_i as well and hence holds for all x, as claimed. □