Chapter 2. Solutions of Equations in One Variable

2.1 The Bisection Method

Note. In this section we iteratively cut an interval in half to approximate the solution to an equation involving a continuous function.

Note. Suppose f is continuous on the interval $[a, b]$ with $f(a)$ and $f(b)$ of opposite signs. By the Intermediate Value Theorem, there is $p \in (a, b)$ such that $f(p) = 0$. The Intermediate Value Theorem is stated in Section 1.11 as:

Theorem 1.11. Intermediate Value Theorem.
If $f \in C[a, b]$ and K is any number between $f(a)$ and $f(b)$, then there exists a number c in (a, b) for which $f(c) = K$.

You see the Intermediate Value Theorem first in Calculus 1 (MATH 1910); see my online Calculus 1 notes on Section 2.5. Continuity (see Theorem 2.11). A proof is given in Analysis 1 (MATH 4217/5217); see my online Analysis 1 notes on Section 4.1. Limits and Continuity (see Corollary 4-9). Now the Intermediate Value Theorem gives the existence of number c, but it says nothing about how to find it.

Definition. For f a real-valued function defined on a subset of $\mathbb{R}$, a value p such that $f(p) = 0$ is called a *zero* of f.

Note. Burden, Fairs, and Burden refer to a zero of a function also as a "root" of the function. This terminology is common when dealing the a polynomial function, but the term "zero" is more often used to indicate a value where any type of function (not just a polynomial function) equals zero.

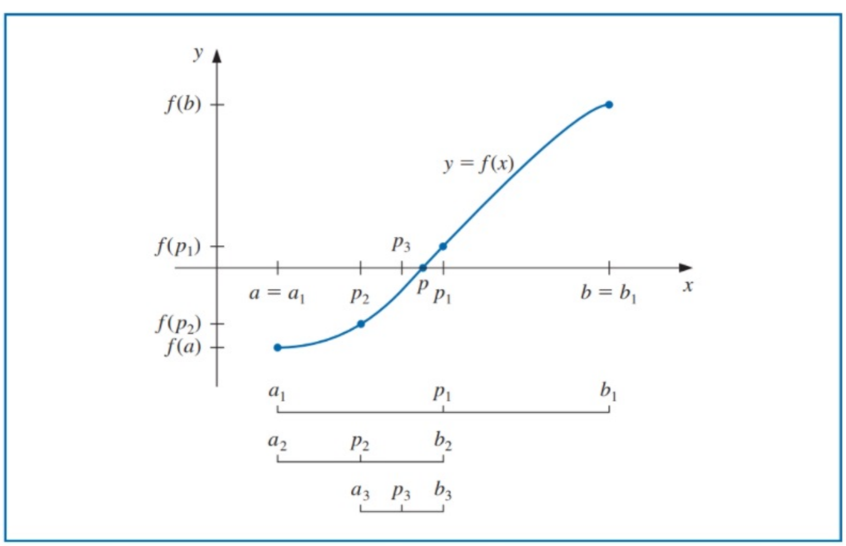
Note. Under the conditions hypothesized in the Intermediate Value Theorem, we repeatedly cut intervals in half (i.e., we *bisect* intervals) in search of the location of the zero of a continuous function. Starting with the interval $[a_1, b_1] = [a, b]$ where $f(a_1)$ and $f(b_1)$ have opposite signs, we consider the midpoint $p_1 = (a_1 + b_1)/2$. If $f(p_1) = 0$ then we take $p = p_1$, otherwise $f(p_1) \neq 0$ and we use the sign of $f(p_1)$ to make a decision:

(1) If $f(p_1)$ and $f(a_1)$ have the same sign, then set $a_2 = p_1$ and set $b_2 = b_1$.

(2) if $f(p_1)$ and $f(a_1)$ have opposite signs, then set $a_2 = a_1$ and set $b_2 = p_1$.

Next, we let $p_2 = (a_1 + b_1)/2$ and iterate the process. This is illustrated in Figure 2.1 in which case $f(p_1)$ and $f(a_1)$ have opposite signs so that $a_2 = a_1$ and $b_2 = p_1$; next, $f(p_2)$ and $f(a_1)$ have the same sign so that $a_3 = p_2$ and $b_3 = b_2$; then (not in Figure 2.1) $f(p_3)$ has the same sign as a_3 so that $a_4 = p_3$ and $b_4 = b_3$.

Figure 2.1



Note/Definition. The *Bisection Method* is given algorithmically as follows. To find a solution to $f(x) = 0$ for continuous function f on the interval $[a, b]$, where $f(a)$ and $f(b)$ have opposite signs:

`INPUT:` endpoints a, b; tolerance TOL; maximum number of iterations N_0.

`OUTPUT` approximate solution p or message failure.

`Step 1.` Set $i = 1$;

$FA = f(a)$.

`Step 2.` While $i \leq N_0$ do Steps 3–6.

`Step 3.` Set $p = 1 + (b - a)/2$;

$FP = f(p)$.

`Step 4.` If $FP = 0$ or $(b - a)/2 < TOL$ then

OUTPUT(p)

STOP.

`Step 5.` Set $i = i + 1$.

`Step 6.` If $FA \cdot FP > 0$ then set $a = p$

$FA = FP$

else set $b = p$.

`Step 7.` OUTPUT (‘Method failed after N_0 iterations, $N_0 =$’ N_0)

STOP.

Note. The stopping criteria of the Bisection Method is given either by the parameter N_0 (since the algorithm ends after N_0 steps, by `Step 2`) or by the tolerance TOL (since the algorithm ends after $(b - a)/2 < TOL$, by `Step 4`). Alternatives to these stopping criteria include considering the relative error $|p_n - p_{n-1}|/|p_n|$

(where $p_n \neq 0$), which could be required to be less than some given tolerance. Burden, Faires, and Burden mention stopping conditions related to making $|p_n - p_{n-1}|$ less than a given tolerance (though they observe that a sequence can satisfy this condition and yet not be convergent; the harmonic series behaves like this and is mentioned in Exercise 2.1.19). They also mention making $|f(p_n)|$ less than a certain tolerance (though this makes the value of f "close to" 0 as desired, it does not guarantee that p_n is close to p, as is desired; Exercise 2.1.20 illustrates this by considering the function $f(x) = (x-1)^{10}$).

Example 2.1.1. Show that $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in the interval $[1, 2]$ and use the Bisection Method to determine an approximation to the root that is accurate to within 10^{-4}.

Solution. First, f is a polynomial function and so is continuous on its domain $\mathbb{R}$. Since $f(1) = -5 < 0$ and $f(2) = 14 > 0$, then the Intermediate Value Theorem implies that f has a zero in $[1, 2]$. Notice that you probably don't know how to algebraically find the root of f in $[1, 2]$. We start the bisection method with $p_1 = 1.5$. Since $f(1.5) = 2.375 > 0$, then we take the left-hand half of $[1, 2]$ to get $[1, 1.5]$ and $p_2 = 1.25$. Next, $f(1.25) = -1.796875 < 0$, so we take the right-hand half of $[1, 1, 5]$ to get $[1.25, 1.5]$ and $p_3 = 1.375$. We now rely on Table 2.1 for the next several values of a_n, b_n, and p_n. Notice that after 13 iterations, $p_{13} = 1.365112305$. Since $p, p_{13} \in [a_{14}, b_{14}]$ then $|p - p_{13}| < |b_{14} - a_{14}| = |1.365234375 - 1.365112305| = 0.000122070$. Also, $|a_{14}| < |p|$ (since $a_{14} > 0$) so that the relative error is

Table 2.1

n	a_n	b_n	p_n	$f(p_n)$
1	1.0	2.0	1.5	2.375
2	1.0	1.5	1.25	−1.79687
3	1.25	1.5	1.375	0.16211
4	1.25	1.375	1.3125	−0.84839
5	1.3125	1.375	1.34375	−0.35098
6	1.34375	1.375	1.359375	−0.09641
7	1.359375	1.375	1.3671875	0.03236
8	1.359375	1.3671875	1.36328125	−0.03215
9	1.36328125	1.3671875	1.365234375	0.000072
10	1.36328125	1.365234375	1.364257813	−0.01605
11	1.364257813	1.365234375	1.364746094	−0.00799
12	1.364746094	1.365234375	1.364990235	−0.00396
13	1.364990235	1.365234375	1.365112305	−0.00194

$$\frac{|p-p_{13}|}{|p|} < \frac{|b_{14}-a_{14}|}{|a_{14}|} = \frac{0.000122070}{1.365112305} \le 9.0 \times 10^{-5} < 1 \times 10^{-4}.$$

To nine decimal places, $p = 1.365230013$, so p_{13} is accurate to three decimal places.

Note. Another application of the Bisection Method can be found in Mathematical Statistics 2 (MATH 4057/5057). It is used in finding confidence intervals in Section 4.3. Confidence Intervals for Parameters of Discrete Distributions.

Note. Burden, Faires, and Burden comment (see page 51):

> "The Bisection method, though conceptually clear, has significant drawbacks. It is relatively slow to converge (that is, N may become quite large before $|p - p_N|$ is sufficiently small), and a good intermediate approximation might be inadvertently discarded. [In fact, in the previous example we see from Table 2.1 that p_9 is closer to the actual value of p that is p_{13}; p_9 is accurate to five decimal places, whereas p_{13} is only accurate to three decimal places.] However, the method has the important property that it always converges to a solution..."

The next theorem puts a bound on the the error used in approximating p with p_n.

Theorem 2.1. Suppose $f \in C[a, b]$ and $f(a) \cdot f(b) < 0$. The Bisection Method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with error

$$|p_n - p| \leq \frac{b-a}{2^n} \text{ when } n \geq 1.$$

Note. Recall the "big oh" notation of Section 1.3. We state Definition 1.18 again.

Definition 1.18 Suppose $\{\beta_n\}_{n=1}^{\infty}$ is a sequence known to converge to zero and $\{\alpha_n\}_{n=1}^{\infty}$ converges to a number α. If a positive constant K exists with $|\alpha_n - \alpha| \leq K|\beta_n|$ for large n, then we say that $\{\alpha_n\}_{n=1}^{\infty}$ converges to α with *rate, or order, of convergence* $O(\beta_n)$. (This expression is read "big oh of β_n".) It is indicated by writing $\alpha_n = \alpha + O(\beta_n)$.

This idea is encountered in Calculus 2 (MATH 1920). In my online notes for Calculus 2 on Section 7.4. Relative Rates of Growth, we have the following.

Definition. Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is *of at most order of* g as $x \to \infty$ if there is a positive integer M for which $f(x)/g(x) \leq M$ for x sufficiently large. We indicate this by writing $f = O(g)$ ("f is big-oh of g).

Since Theorem 2.1 gives $|p_n - p| \leq \dfrac{b-a}{2^n}$ when $n \geq 1$, we have that the sequence $\{p_n\}_{n=1}^{\infty}$ converges to p with the rate of convergence $O(1/2^n)$; that is, $p_n = p + O(1/2^n)$.