2.2. Fixed-Point Iteration

Note. In this section we give an algorithm to find a fixed point of a function. The algorithm does not work for all functions, but works for functions which are "contractions." We also relate fixed points of certain functions to zeros of others (and conversely).

Definition. A number p is a *fixed point* for a given function g is $g(p) = p$.

Note. Function f has a fixed point at $x = p$ is and only if function $g(x) = f(x) - x$ has a zero of $x = p$. Therefore we can use a technique for finding the zeros of a function (such as the Bisection Method of the previous section) to find fixed points of a function, *or* we can use a technique for finding the fixed points of a function (like fixed-point iteration of this section) to find the zeros of a function.

Note. A function may have no fixed points (for example, $f(x) = x + 1$) or it may have multiple fixed points (for example $f(x) = x^3$ has fixed points $x = -1$, $x = 0$, and $x = 1$). To successfully apply a numerical technique, we need to know that a fixed point exists. We will consider the cases where a unique fixed point exists and we will give a technique that is guaranteed to find this fixed point. This leads us to the following result.

Theorem 2.3.

(i) If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has at least on fixed point in $[a,b]$.

(ii) If, in addition, $g'(x)$ exists on (a,b) and a positive constant $k < 1$ exists with $|g'(x)| \leq k$ for all $x \in (a,b)$, then there is exactly on fixed point in $[a,b]$.

Note. Figure 2.3 illustrates a situation in which Theorem 2.3 applies. Notice on the interval $[a,b]$ that function g has a unique fixed point $x = p$ and that function g has tangent lines on $[a,b]$ with slopes between -1 and 0 (so that $|g'(x)| < 1$ on (a,b)).

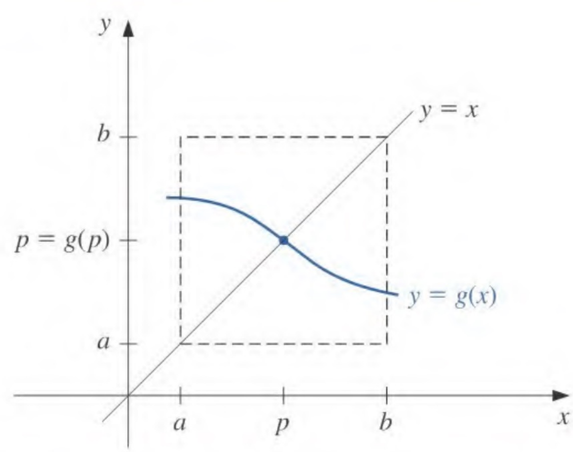

Figure 2.3

Example 2.2.2. Show that $g(x) = (x^2 - 1)/3$ has a unique fixed point on the interval $[-1,1]$.

Solution. First, g is continuous on $[-1,1]$, so by the Extreme Value Theorem (see my online Calculus 1 [MATH 1910] notes on Section 4.1. Extreme Values of Functions on Closed Intervals; notice Theorem 4.1) g has a maximum and a minimum on $[-1, 1]$. By the Local Extreme Values Theorem (see Theorem 4.2 in the section of Calculus 1 notes just mentioned), the extrema of g on $[-1, 1]$ occur either at critical points in $[-1, 1]$ or at the endpoints -1 and 1. Since $g'(x) = 2x/3$ then $x = 0$ is the only critical point, and $g(-1) = 0$, $g(0) = -1/3$, and $g(1) = 0$. Hence the maximum of g on $[-1, 1]$ is 0 and the minimum is $-1/3$. So for all $x \in [-1, 1]$ we have $g(x) \in [-1/3, 0] \subset [-1, 1]$. Therefore, by Theorem 2.3(i) we know that g has at least one fixed point in $[-1, 1]$. In addition, $|g'(x)| = |2x/3| \leq 2/3 = k < 1$ for all $x \in [-1, 1]$, so by Theorem 2.3(ii) the fixed point of g in $[-1, 1]$ is unique. $\square$

Note. We can find the fixed points of $g(x) = (x^2 - 1)/3$ from Example 2.2.2 using the quadratic equation. Setting $g(p) = (p^2 - 1)/3 = p$ we have $p^2 - 3p - 1 = 0$ and so $p = \dfrac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2(1)} = \dfrac{3 \pm \sqrt{13}}{2}$. So the unique fixed point of g in $[-1, 1]$ is $p = (3 - \sqrt{13})/2 \approx -0.3028$.

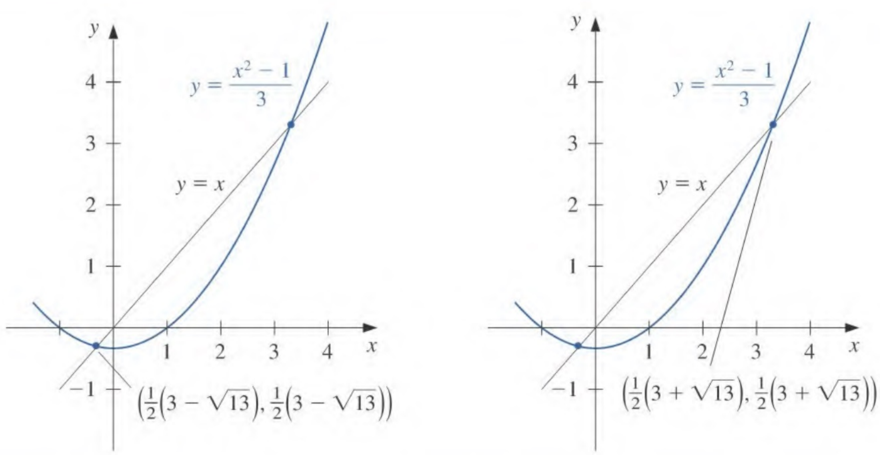

Figure 2.4

Notice that $p = (3+\sqrt{13})/2 \approx 3.3028 \in [3, 4]$, so that g also has a fixed point in the interval $[3, 4]$. Notice that $g(4) = 5$ so that $g(x)$ is not in $[3, 4]$ for all $x \in [3, 4]$ (so the hypotheses of Theorem 2.3(i) are not satisfied). Also $|g'(4)| = |2(4)/3| = 8/3 > 1$, so the hypotheses of Theorem 2.3(ii) are not satisfied either! Yet, g still has a fixed point on $[3, 4]$. See Figure 2.4. That is, we have that the hypotheses of Theorem 2.3 are sufficient to guarantee a unique fixed point, but they are not necessary for the existence of a unique fixed point.

Note. We are interested in approximating a fixed point of continuous function g. The technique of *fixed-point iteration* is based on the sequence $\{p_n\}$ with initial approximation p_0 and p_n inductively defined as $p_n = g(p_{n-1})$ for $n \geq 1$. If the sequence converges to p and g is continuous, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g\left(\lim_{n\to\infty} p_{n-1}\right) = g(p),$$

and $x = p$ is a fixed point of g. The technique is illustrated in Figure 2.6. Below in Theorem 2.4 we will see a condition on g that insures the sequence $\{p_n\}$ converges to a fixed point p of g. First we state the technique as an algorithm.

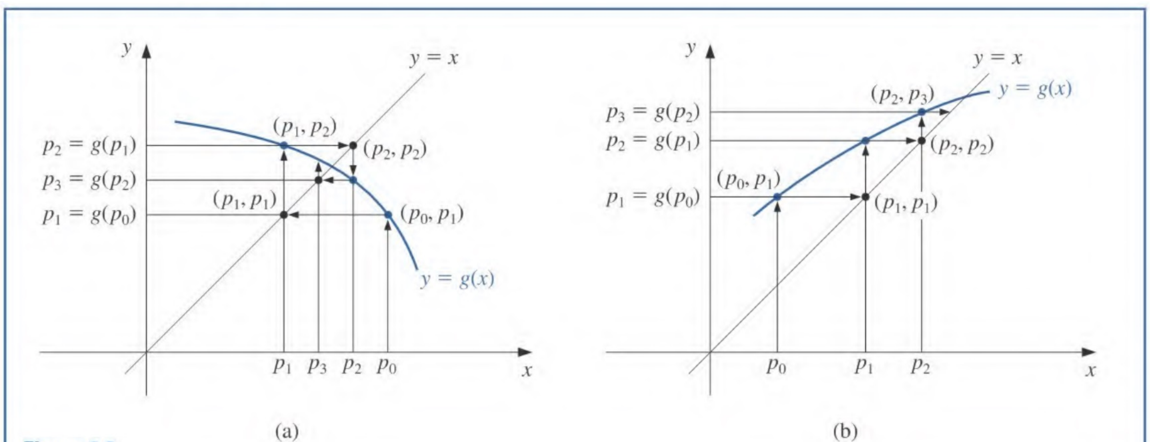

Figure 2.6

Algorithm 2.2. The *Fixed-Point Iteration* method is given algorithmically as follows. To find a solution to $p = g(p)$ given an initial approximation p_0:

`INPUT:` initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

`OUTPUT` approximate solution p or message of failure.

`Step 1.` Set $i = 1$.

`Step 2.` While $i \leq N_0$ do Steps 3–6.

`Step 3.` Set $p = g(p_0)$. (*Compute* p_i.)

`Step 4.` If $|p - p_0| < TOL$ then

OUTPUT(p); (*The procedure was successful.*)

STOP.

`Step 5.` Set $i = i + 1$.

`Step 6.` Set $p_0 = p$. (*Update* p_0.)

`Step 7.` OUTPUT (‘Method failed after N_0 iterations, $N_0 =$’ N_0)

STOP.

Theorem 2.4. Fixed-Point Theorem. Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose, in addition, that g' exists on (a, b) and that a constant $0 < k < 1$ exists with $|g'(x)| \leq k$ for all $x \in (a, b)$. Then for any number $p_0 \in [a, b]$, the sequence defined by $p_n = g(p_{n-1})$, for $n \in \mathbb{N}$, converges to the unique fixed point $p \in [a, b]$.

Corollary 2.5. If g satisfies the hypotheses of Theorem 2.4, then bounds for the error involved in using p_n to approximate p are given by $|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$ and $|p_n - p| \leq \dfrac{k^n}{1-k}|p_1 - p_0|$ for all $n \in \mathbb{N}$.

Note. We see from Corollary 2.5 that if $|g'(x)| \leq k$ where k is "small," then the convergence of the sequence $\{p_n\}$ will be fast and we can get better approximations with fewer iterations of Algorithm 2.2. One way to minimize k is to consider g on a smaller interval $[c, d] \subset [a, b]$; however, this requires some up-front knowledge of location of fixed point p.

Note. The Fixed-Point Theorem (Theorem 2.4) is based on the general Contraction Mapping Theorem. The condition $|g'(x)| \leq k < 1$ on $[a, b]$ implies that g is a "contraction" on $[a, b]$. A contraction is a function that brings any two points closer together (in our context, this means $|g(x_1) - g(x_2)| < |x_1 - x_2|$ for all $x_1, x_2 \in [a, b]$). The Contraction Mapping Theorem is covered in Fundamentals of Functional Analysis (MATH 5740), though specific applications are not presented in that class. See my online notes on Section 2.12. Fixed Points and Contraction Mappings; notice Theorem 2.44. It is also covered in Real Analysis 2 (MATH 5220) in the setting of metric spaces where it is called the Banach Contraction Principle; see my online notes for Real Analysis 2 on Section 10.3. The Banach Contraction Principle; notice Theorem 10.3.B. It may also be covered in Applied Math 1 (MATH 5610). See my online notes for Applied Math 1 (these are the version of the notes used in the fall 1996 class) on Section 3.3. The Contraction Mapping Theorem; notice Theorem 3.3.1.

Example 2.2.A. (Based on the "Illustration" on page 60.) The equation $x^3 + 4x^2 - 10 = 0$ has a unique root in $[1, 2]$. We rearrange this equation in such a way as to use Fixed-Point Iteration to solve the equation. Notice that $x^3+4x^2-10=0$ is equivalent to $x^2(x+4)=10$ or $x^2=10/(x+4)$ or, since we are interested in a positive solution, $x=\left(\frac{10}{x+4}\right)^{1/2}$. We can then solve this equation by finding a fixed point of $g_4(x)=\left(\frac{10}{x+4}\right)^{1/2}=\sqrt{10}(x+4)^{-1/2}$ (notice that Burden, Faires, and Burden consider five different approaches to this problem, some of which work and some of which do not). Notice that $g_4'(x)=\sqrt{10}\left(\frac{-1}{2}\right)(x+4)^{-3/2}=\frac{-\sqrt{10}}{2(x+4)^{3/2}}$ and for $x\in[1,2]$ we have $|g_4'(x)|=\left|\frac{-\sqrt{10}}{2(x+4)^{3/2}}\right|\leq\frac{\sqrt{10}}{2(5)^{3/2}}\approx 0.1414$. Therefore $|g_4'(x)|<0.15$ for $x\in[1,2]$. Also, $g_4'(x)<0$ for $x\in[1,2]$, so g_4 is decreasing on $[1,2]$ and the maximum of g_4 on this interval is $g_4(1)=\sqrt{10}((1)+4)^{-1/2}=\sqrt{2}\approx 1.4142$ and the minimum is $g_4(2)=\sqrt{10}((2)+4)^{-1/2}=\sqrt{5/3}\approx 1.2910$. Therefore $g(x)\in[1,2]$ for $x\in[1,2]$, and the hypotheses of the Fixed-Point Theorem (Theorem 2.4) are satisfied, so we know that the sequence $\{p_n\}$ converges to the unique fixed point of g_4 (and hence the unique solution to the original equation) in $[1,2]$. Suppose we take $TOL=0.00001$ and $p_0=1.5$ in the Fixed-Point Iteration algorithm. We get the following values:

n	p_n	$\lvert p_n-p_{n-1}\rvert$
0	1.5	-
1	1.348399725	0.151600275
2	1.367376372	0.018976647
3	1.364957015	0.002419357
4	1.365264748	0.000307733
5	1.365225594	0.000039154
6	1.365230576	0.000004983

So after 6 iterations the Fixed-Point Iteration algorithm terminates (in `Step 4`) and outputs 1.365230576, which we know is within $0.000005 < 0.00001 = TOL$ of the actual value of the fixed point. Nine more iterations (for a total of 15) gives the fixed point to 10 decimal places (see Burden, Faires, and Burden's Table 2.2).

Revised: 12/17/2022