Chapter 3. Interpolation and Polynomial Approximation

3.1 Interpolation and the Lagrange Polynomial

Note. In this section we define the Lagrange polynomial of degree n that passes through $n+1$ given points and discuss its use for interpolation between the given points.

Note. Recall from Section 1.1 that the set of functions continuous on interval $[a,b]$ is denoted $C[a,b]$; more generally, the set of functions continuous on set X is denoted $C(X)$. The set of functions which have a continuous nth derivative on interval $[a,b]$ is denoted $C^n[a,b]$; or more generally $C^n(X)$. For details on these and related classes of functions, see my supplemental online notes for Complex Analysis 1 (MATH 5510) on A Primer on Lipschitz Functions.

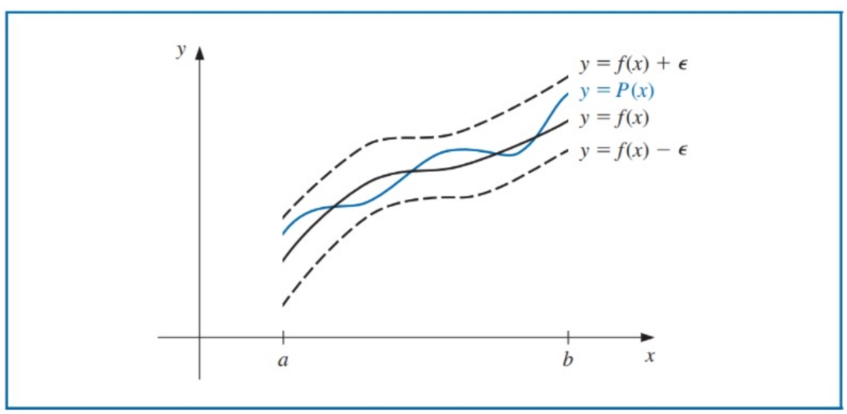
Definition. An *algebraic polynomial* or *polynomial function* is a function mapping $\mathbb{R} \to \mathbb{R}$ of the form $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot + a_2 x^2 + a_1 x + a_0$, where n is a nonnegative integer and $a_0, a_1, \ldots, a_n \in \mathbb{R}$ with $a_n \neq 0$. The real constants $a_0, a_1, \ldots, a_n$ are the *coefficients* of the polynomial and n is the degree.

Note. Polynomial functions involve elementary computation, only requiring multiplication and addition. So it would be computationally convenient to approximate more complicated functions (such as trigonometric or exponential functions) with polynomial functions. The Weierstrass Approximation Theorem tells us that, on certain kinds of sets, these approximations exist to any (nonzero) level accuracy desired. More precisely, we have the following.

Theorem 3.1. Weierstrass Approximation Theorem.
Suppose f is defined and continuous on $[a, b]$. For each $\varepsilon > 0$, there exists a polynomial $P(x)$ with the property that

$$|f(x) = P(x)| < \varepsilon \text{ for all } x \in [a, b].$$

Figure 3.1



Note. Anton R. Schep of the University of South Carolina has a nice, concise (2 page) and self-contained proof of the Weierstrass Approximation Theorem posted online (accessed 3/14/2022). Dr. Schep's proof is essentially the proof of Weierstrass, which appeared originally in "Uber die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen," *Sitzungsberichte der Kniglich Preuischen Akademie der Wissenschaften zu Berlin*, 1885 (11). You might see the Weierstrass Approximation Theorem in a senior-level class on Introduction to Applied Math (Such as ETSU's MATH 4027/5027) or a class on approximation theory. It can also be covered in a class covering functional analysis. For example, it may be covered in Real Analysis sequence (MATH 5210/5220) in Section 12.3. The Stone-Weierstrass Theorem. The following is proved, which is a generalization of Theorem 3.1.

The Stone-Weierstrass Approximation Theorem. Let X be a compact Hausdorff space. Suppose $\mathcal{A}$ is an algebra of continuous real-valued functions on X that separates points in X and contains the constant functions. Then $\mathcal{A}$ is dense in the space of continuous functions $C(X)$.

Definition. Let (x_0, y_0) and (x_1, y_1) be two points in the Cartesian plane $\mathbb{R}^2$ where $x_0 \neq x_1$. Define the functions

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \text{ and } L_1(x) = \frac{x - x_0}{x_1 = x_0}.$$

The *linear Lagrange interpolating polynomial* through points (x_0, y_0) and (x_1, y_1) is

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1) = \frac{x - x_1}{x_0 - x_1}f(x_0) + \frac{x - x_0}{x_1 - x_0}f(x_1),$$

where $y_0 = f(x_0)$ and $y_1 = f(x_1)$.

Note. The linear Lagrange interpolating polynomial P is a first degree polynomial function. When $x = x_0$ we have

$$P(x_0) = \frac{(x_0) - x_1}{x_0 - x_1}f(x_0) + \frac{(x_0) - x_0}{x_1 - x_0}f(x_1) = (1)f(x_0) + (0)f(x_1) = f(x_0) = y_0,$$

and

$$P(x_1) = \frac{(x_1) - x_1}{x_0 - x_1} f(x_0) + \frac{(x_1) - x_0}{x_1 - x_0} f(x_1) = (0) f(x_0) + (1) f(x_1) = f(x_1) = y_1.$$

So, $y = P(x)$ is a function whose graph is a line that passes through the points (x_0, y_0) and (x_1, y_1). Of course we have an easier way of determining this line, but we introduce this idea to motivate a generalization below.

Note. Burden, Faires, and Burden give the following as Theorem 3.2 (in which they make a uniqueness claim based on the degree of the polynomial function), but we state it here as a definition.

Definition. Let $x_0, x_1, \ldots, x_n$ be $n + 1$ distinct numbers and f a function whose values are given at these numbers. The *nth Lagrange interpolating polynomial* through the $n + 1$ points $(x_i, f(x_i))$ for $i = 0, 1, 2, \ldots, n$ is

$$P(x) = f(x_0)L_{n,0}(x) + f(x_1)L_{n,1}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x),$$

where for $k = 0, 1, \ldots, n$,

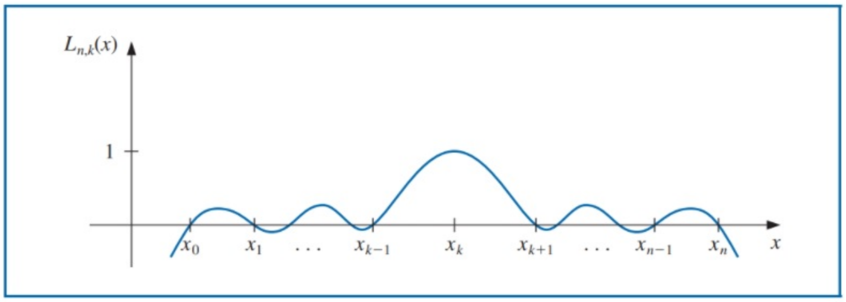
$$\begin{aligned} L_{n,k}(x) &= \frac{(x - x_0)(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_0)(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)} \\ &= \prod_{i=0, i \neq k}^{n} \frac{(x - x_i)}{(x_k - x_i)}. \end{aligned}$$

We write $L_{n,k}(x)$ as $L_k(x)$ when there is not confusion as to its degree.

Note. For $n = x_k$ we have $L_{n,k}(x_k) = \prod_{i=0, i \neq k}^{n} \frac{(x_k - x_i)}{(x_k - x_i)} = 1$, and for $x = x_j$ where

$j \in \{0, 1, 2, \ldots, n\}$ but $j \neq k$, we have $L_{n,k}(x_j) = \displaystyle\prod_{i=0, i\neq k}^{n} \frac{(x_j - x_i)}{(x_k - x_i)} = 0$ (since $(x_j - x_i) = 0$ when $i = j$). So $P(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$, as desired. Figure 3.5 shows the graph of a "typical" $L_{n,k}$ where n is even.

Figure 3.5



Example 3.1.2. **(a)** Use the numbers $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = 1/x$. **(b)** Use this polynomial to approximate $f(3) = 1/3$.

Solution. **(a)** By definition, we have (skipping some arithmetic steps):

$$
\begin{aligned}
L_0(x) &= \frac{(x - x_1)(x - x_2)}{x_0 - x_1)(x_0 - x_2)} = \frac{(x - 2.75)(x - 4)}{(2 - 2.75)(2 - 4)} = \frac{2}{3}(x - 2.75)(x - 4), \\
L_1(x) &= \frac{(x - x_0)(x - x_2)}{x_1 - x_0)(x_1 - x_2)} = \frac{(x - 2)(x - 4)}{(2.75 - 2)(2.75 - 4)} = -\frac{16}{15}(x - 2)(x - 4), \\
L_2(x) &= \frac{(x - x_0)(x - x_1)}{x_2 - x_0)(x_2 - x_1)} = \frac{(x - 2)(x - 2.75)}{(4 - 2)(4 - 2.75)} = \frac{2}{5}(x - 2)(x - 2.75).
\end{aligned}
$$

Since $f(x_0) = f(2) = 1/2$, $f(x_1) = f(2.75) = 4/11$, and $f(x_2) = f(4) = 1/4$, then the second Lagrange interpolating polynomial based on these function values is (simplifying and skipping arithmetic steps):

$$
P(x) = \sum_{k=0}^{n} f(x_k) L_k(x)
$$

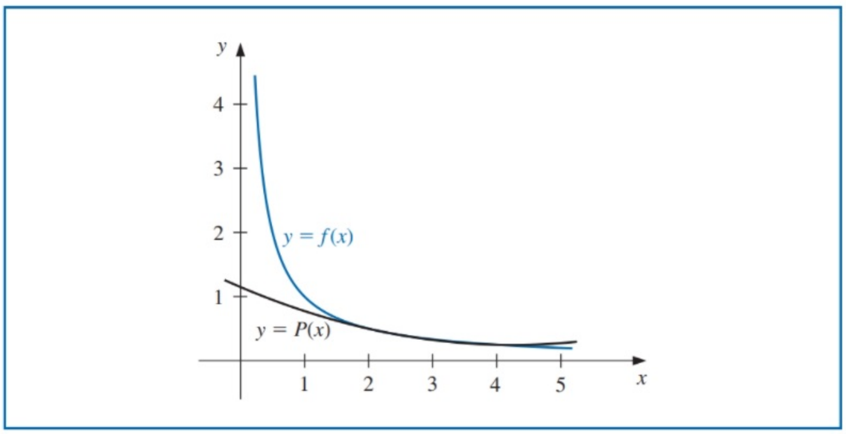
$$= \frac{1}{3}(x-2.75)(x-4) - \frac{64}{165}(x-2)(x-4) + \frac{1}{10}(x-2)(x-2.75)$$
$$= \boxed{\frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}}.$$

(b) To approximate $f(3)$, we consider

$$P(3) = \frac{1}{22}(3)^2 - \frac{35}{88}(3) + \frac{49}{44} = \frac{9}{22} - \frac{105}{88} + \frac{49}{44} = \frac{28}{88} \approx \boxed{0.32955}.$$

So this is a reasonable approximation to $1/3$ (especially considering the fact that we estimated with just a second degree polynomial; notice that we are interpolating here since 3 lies within the range of the given function values, and it's "close" to one of the known function values, namely 2.75). Figure 3.6 gives a graph of f and P together, showing the close agreement between the two near the data points. □

Figure 3.6



Note. Another application of Lagrange polynomials can be found in Elementary Number Theory (MATH 3120). A Lagrange polynomial of degree n can be used to generate the first $n+1$ prime numbers. See my online notes on Section 22. Formulas for Primes.

Note. We need the Generalized Rolle's Theorem for the proof of the next theorem.

This is stated in Section 1.1, but we restate it here:

Theorem 1.10. Generalized Rolle's Theorem.

Suppose $f \in C[a,b]$ is n times differentiable on (a,b). If $f(x) = 0$ at the $n+1$ distinct numbers $a \leq x_0 < x_1 < \cdots < x_n \leq b$, then a number $c \in (x_0, x_n)$ and hence in (a,b) exists with $f^{(n)}(c) = 0$.

Note. We are now ready to relate a function f in $C^{n+1}[a,b]$ to its nth Lagrange polynomial P. In so doing, we introduce an "error term," which allows us to gauge the level of accuracy we have in approximating f with P on the interval $[a,b]$.

Theorem 3.3. Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a,b]$ and $f \in C^{n+1}[a,b]$. Then for each x in $[a,b]$, a number $\xi(x)$ between $\min\{x_0, x_1, \ldots, x_n\}$, and the $\max\{x_0, x_1, \ldots, x_n\}$ and hence in (a,b), exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n),$$

where $P(x)$ is the nth Lagrange interpolating polynomial.

Note. Theorem 3.3 may remind you of Taylor's Theorem from Section 1.1, Calculus 2 (MATH 1920), or Analysis 1 (MATH 4217/5217). However, Taylor's Theorem involves matching the *derivatives* of function f with the derivatives of polynomial P, and then considering a remainder term (instead of matching *values* of function f, as we do with Lagrange interpolating polynomials). These are closely related ideas, and in Exercise 3.1.22 you are to show that Taylor's Theorem can be proved from Theorem 3.3. For the statement of Taylor's Theorem and its use in Calculus 2, see my online notes on Section 10.9 Convergence of Taylor Series (notice Taylor's Theorem, Taylor's Formula, and the Remainder Estimation Theorem). For Analysis 1 material, see my online notes on Section 5.2. Some Mean Value Theorems (notice Taylor's Theorem and Taylors Theorem Alternative Version). The statement of Taylor's Theorem from Section 1.1 of Burden, Faires, and Burden is:

Theorem 1.14. Taylor's Theorem.

Suppose $f \in C^n[a,b]$, $f^{(n+1)}$ exists on $[a,b]$, and $x_0 \in [a,b]$. For every $x \in [a,b]$ there exists a number $\xi(x)$ between x_0 and x with $f(x) = P_n(x) + R_n(x)$ where

$$\begin{aligned} P_x(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k, \text{ and} \\ R_n(x) &= \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1}. \end{aligned}$$

Example 3.1.3. We conclude this section with an illustration of the use of the "error term" of Theorem 3.3 Determine the error term for the polynomial of Example 3.1.2 and the maximum error when the polynomial is used to approximate $f(x) = 1/x$ for $x \in [2,4]$.

Solution. In Example 3.1.2 we found the second Lagrange interpolating polynomial for $f(x) = 1/x = x^{-1}$ based on the x-values 2, 2.75, and 4. By Theorem 3.3 we would consider this an approximation on the interval $[\min\{2, 2.75, 4\}, \max\{2, 2.75, 4\}] = [2,4]$. To find how big the error term could be on this interval, we consider $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, and $f'''(x) = -6x^{-4}$. The error term for the second Lagrange interpolating polynomial is

$$\frac{f'''(\xi x))}{3!}(x-x_0)(x-x_1)(x-x_2) = \boxed{-(\xi(x))^{-4}(x-2)(x-2.75)(x-4)}$$

where $\xi(x) \in (a,b)$. Since $(\xi(x))^{-4}$ reaches its maximum for $\xi(x) \in [2,4]$ when $\xi(x) = 2$, then we have a bound on $(\xi(x)^{-4}$ of $2^{-4} = 1/16$. Now the polynomial part of the error term is

$$g(x) = (x-2)(x-2.75)(x-4) = x^3 - \frac{35}{4}x^2 + \frac{49}{2}x - 22,$$

and this has derivative $g'(x) = 3x^2 - \frac{35}{2}x + \frac{49}{2} = \frac{1}{2}(3x-7)(2x-7)$, so that the critical points are $x = 7/3$, where $g(7/3) = 25/108$, and $x = 7/2$, where $g(7/2) = -9/16$. We also check the endpoints of $[a,b] = [2,4]$ and find that $g(2) = g(4) = 0$ (since the endpoints are x-values used in the determination of g), so that the maximum of g on $[2,4]$ is $25/108$ and the minimum is $-9/16$. However, we are interest in how big the *absolute value* of g is (and how big the absolute value of the error term is), we have have that $|g(x)| \le 9/16$ for $x \in [2,4]$. So the error term itself is bounded by

$$\left|\frac{f'''(\xi x))}{3!}(x-x_0)(x-x_1)(x-x_2)\right| = |-(\xi(x))^{-4}(x-2)(x-2.75)(x-4)|$$

$$\le \frac{1}{16}\frac{9}{16} = \boxed{\frac{9}{256}}.$$

Since $9/256 \approx 0.03516$, then we see that the error made in approximating $f(3) = 1/3$ with 0.32955, as we did in Example 3.1.2, is $|1/3 - 0.32955| \approx 0.00378 < 0.03516$, as expected. □

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