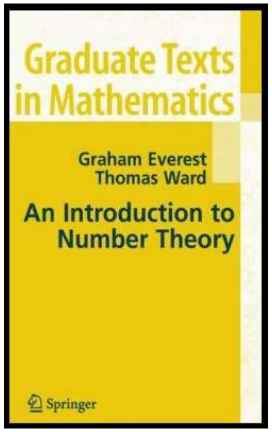
Number Theory

Section 1.1. Euclid and Primes—Proofs of Theorems



Theorem 1.2

Theorem 1.2. Euclid's Infinite Primes Theorem. There are infinitely many primes.

Proof. ASSUME there are only finitely many primes, say $p_1, p_2, \ldots, p_r$. Let $N = p_1 p_2 \cdots p_r + 1 > 1$. By the Fundamental Theorem of Arithmetic (Theorem 1.1), N can be expressed as a product of prime numbers and so is divisible by some prime p_k in the list $p_1, p_2, \ldots, p_r$. Since p_k also divides $p_1 p_2 \cdots p_k \cdots p_r$, then p_k divides the difference $N - p_1 p_2 \cdots p_r = 1$. But $p_k > 1$ cannot divide 1, a CONTRADICTION. So the assumption that there are only finitely many primes is false and hence there are infinitely many primes, as claimed. □

Theorem 1.2

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Proof. ASSUME there are only finitely many primes, say $p_1, p_2, \ldots, p_r$. Consider the product $X = \prod_{k=1}^{r} \left(1 - \frac{1}{p_k}\right)^{-1}$. Since this is a finite product, the X is finite. Notice that each term can be written as the sum of a convergent geometric series:

$$\frac{1}{1 - \frac{1}{p}} = 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots.$$

Notice that for any fixed $K \in \mathbb{N}$, we have

$$\frac{1}{1 - \frac{1}{p}} \geq 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^K}.$$

Theorem 1.2 (continued 1)

Proof (continued). Substituting the previous inequality into the representation of X we have

$$\begin{aligned} X &\geq \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^K}\right) \cdot \left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^K}\right) \\ &\quad \cdot \left(1 + \frac{1}{5} + \frac{1}{5^2} + \cdots + \frac{1}{5^K}\right) \cdots \left(1 + \frac{1}{p_r} + \frac{1}{r_r^2} + \cdots + \frac{1}{p_r^K}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^K 3^K 5^K \cdots p_r^K} \\ &= \sum_{n \in \mathcal{N}(K)} \frac{1}{n}, \end{aligned}$$

where $\mathcal{N}(K) = \{n \in \mathbb{N} \mid n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}, 0 \leq e_i \leq K \text{ for all } i\}$. Notice that by the Fundamental Theorem of Arithmetic, the elements of $\mathcal{N}(K)$ are distinct.

Theorem 1.2 (continued 2)

Theorem 1.2. Euclid's Infinite Primes Theorem. There are infinitely many primes.

Proof (continued). Given an $n \in \mathbb{N}$, if K is large enough (and under our assumption of a finite number of primes), we have that $X \geq \sum_{n=1}^{\infty} \frac{1}{n}$. Since the harmonic series diverges to infinity, the X must be infinity, a CONTRADICTION. So the assumption that there are only finitely many primes is false and hence there are infinitely many primes, as claimed. □