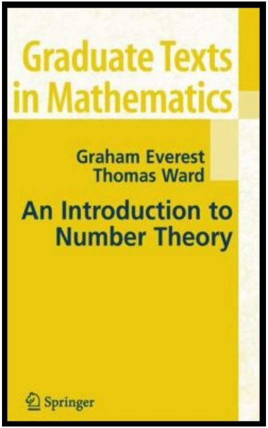
# Number Theory

**Section 1.2. Summing over the Primes**—Proofs of Theorems



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$$P = \sum_{t=1}^{\infty} \left( \sum_{p > N} \frac{1}{p} \right)^t < \sum_{t=1}^{\infty} \frac{1}{2^t} = 1.$$

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$$P = \sum_{t=1}^{\infty} \left( \sum_{p > N} \frac{1}{p} \right)^t < \sum_{t=1}^{\infty} \frac{1}{2^t} = 1.$$

# Theorem 1.3 (continued 1)

**Proof (continued).** Now if the expression of $1 + nQ$ as a product of primes is $1 + nQ = q_1^{e_1} q_2^{e_2} \cdots q_r^{e_r}$, then the expression $\left( \sum_{p>N} \frac{1}{p} \right)^t$ contains each $1/q_i^{e_i}$ for each $1 \leq i \leq r$ (when $p = q_i > N$ and $t = e_i$). Next, the expression $\sum_{t=1}^{\infty} \left( \sum_{p>N} \frac{1}{p} \right)^t$ contains a term of the form $\frac{s}{q_1^{e_1} q_2^{e_2} \cdots q_r^{e_r}} = \frac{s}{1 + nQ}$ where $s \geq 1$ (by adding the previous terms and getting a common denominator). So the left hand side of

$$P = \sum_{t=1}^{\infty} \left( \sum_{p>N} \frac{1}{p} \right)^t < \sum_{t=1}^{\infty} \frac{1}{2^t} = 1$$

contains a term greater than or equal to $1/(1 + nQ)$ for each $n \in \mathbb{N}$.

# Theorem 1.3 (continued 2)

**Proof (continued).** Therefore

$$\sum_{n=1}^{\infty} \frac{1}{1+nQ} \leq \sum_{t=1}^{\infty} \left( \sum_{p>N} \frac{1}{p} \right)^t < 1.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+nQ}$ is a bounded positive term series, then it converges. But since $1 + nQ \leq 2nQ$ then $\displaystyle\sum_{n=1}^{K} \frac{1}{1+nQ} \geq \sum_{n=1}^{K} \frac{1}{2nQ} = \frac{1}{2Q} \sum_{n=1}^{K} \frac{1}{n}$ for any $K$, and since $\displaystyle\sum_{n=1}^{K} \frac{1}{n} \to \infty$ as $K \to \infty$ (since the limit yields the harmonic series), then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+nQ}$ diverges by the Direct Comparison Test.

# Theorem 1.3 (continued 3)

**Theorem 1.3.** The series $\displaystyle\sum_{p\in\mathbb{P}} \frac{1}{p}$ diverges.

**Proof (continued).** But this is a CONTRADICTION to the fact that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+nQ}$ converges. So the original assumption that $\displaystyle\sum_{p\in\mathbb{P}} \frac{1}{p}$ converges is false and hence this series diverges, as claimed. $\square$

# Theorem 1.3

**Theorem 1.3.** The series $\displaystyle\sum_{p\in\mathbb{P}} \frac{1}{p}$ diverges.

**Proof.** Fix $N$ and let

$$\mathfrak{N}(N) = \{n \in \mathbb{N} \mid \text{ all prime factors of } n \text{ are less than or equal to } N\}.$$

Then, as in the second proof of Euclid's Infinite Primes Theorem (Theorem 1.2) we have

$$\sum_{n\in\mathfrak{N}(N)} \frac{1}{p} = \prod_{p\le N}(1 + p^{-1} + p^{-2} + p^{-3} + \cdots) = \prod_{p\le N}(1 - p^{-1})^{-1}.$$

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Now for $n \le N$, then $n \in \mathfrak{N}(N)$, so $\displaystyle\sum_{n\le N} \frac{1}{n} \le \sum_{n\in\mathfrak{N}(N)} \frac{1}{n}$. By the inequality in Note 1.1.A, we have

$$\log N \le \log(N+1) \le \sum_{n\in\mathfrak{N}(N)} \frac{1}{n} = \prod_{p\le N}(1 - p^{-1})^{-1}. \qquad (1.5)$$

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Now for $n \le N$, then $n \in \mathfrak{N}(N)$, so $\displaystyle\sum_{n \le N} \frac{1}{n} \le \sum_{n \in \mathfrak{N}(N)} \frac{1}{n}$. By the inequality in Note 1.1.A, we have

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# Theorem 1.3 (continued 1)

**Proof (continued).** ...By the inequality in Note 1.1.A, we have

$$\log N \leq \sum_{n \in \mathfrak{N}(N)} \frac{1}{n} = \prod_{p \leq N} (1 - p^{-1})^{-1}. \tag{1.5}$$

With $f(v) = (1 - v)\exp(v + v^2)$, we have $f'(v) = v(1 - 2v)\exp(v + v^2)$ and $f'(v) \geq 0$ for $v \in [0, 1/2]$ then $f$ is increasing on this interval. Since $f(0) = 1$ then $f(v) = (1 - v)\exp(v + v^2) \geq 1$ for all $v \in [0, 1/2]$; that is, $\dfrac{1}{1 - v} \leq e^{v + v^2}$ for $v \in [0, 1/2]$. For $p$ prime, we have $v = 1/p \leq 1/2$, so this inequality gives us that

$$\prod_{p \leq N} (1 - p^{-1})^{-1} \leq \prod_{p \leq N} \exp(p^{-1} + p^{-2}).$$

Combining this with Equation (1.5) and taking logarithms gives

$$\log \log N \leq \sum_{p \leq N} (p^{-1} + p^{-2}).$$

# Theorem 1.3 (continued 1)

**Proof (continued).** . . . By the inequality in Note 1.1.A, we have

$$\log N \leq \sum_{n \in \mathfrak{N}(N)} \frac{1}{n} = \prod_{p \leq N} (1 - p^{-1})^{-1}. \tag{1.5}$$

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# Theorem 1.3 (continued 2)

**Theorem 1.3.** The series $\displaystyle\sum_{p\in\mathbb{P}} \frac{1}{p}$ diverges.

**Proof (continued).** Since this is a positive term series then we have

$$\log\log N \le \sum_{p\le N}(p^{-1}+p^{-2}) = \sum_{p\le N}\frac{1}{p} + \sum_{p\le N}\frac{1}{p^2}.$$

As shown by Euler in his solution to the "Basel problem" in 1734, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$ (or we know simply that the series converges since it is a $p$-series with $p = 2 > 1$). That is, $\displaystyle\sum_{p\le N}\frac{1}{p^2}$ is bounded. Since $\log\log N \to \infty$ as $N \to \infty$, then $\displaystyle\sum_{p\le N}\frac{1}{p} \to \infty$ as $N\to\infty$, as claimed. $\square$

# Theorem 1.5

**Theorem 1.5. Euler Product Representation.**
For any real $\sigma$ with $\sigma > 1$, $\zeta(\sigma) = \prod_p (1 - p^{-\sigma})^{-1}$.

**Proof.** For any $\sigma > 1$,

$$(1 - 2^{-\sigma})\zeta(\sigma) = (1 - 2^{-\sigma}) \sum_{n=1}^{\infty} \frac{1}{n^\sigma} = \sum_{n=1}^{\infty} \frac{1}{n^\sigma} - \sum_{n=1}^{\infty} \frac{1}{(2n)^\sigma}$$

$$= \sum_{n \text{ odd}} \frac{1}{n^\sigma} = 1 + \sum_{n \text{ odd}, n>2} \frac{1}{n^\sigma},$$

where summing over all $n$ odd where $n > 2$ is equivalent to summing over all $n$ such that the prime factors of $n$ are greater than 2. That is, in the final term we are summing over all $n$ such that if prime $p$ divides $n$, then $p > 2$. We denote this as

$$(1 - 2^{-\sigma})\zeta(\sigma) = 1 + \sum_{n \text{ odd}, n>2} \frac{1}{n^\sigma} = 1 + \sum_{p \,|\, n \Rightarrow p>2} \frac{1}{n^\sigma}.$$

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# Theorem 1.5 (continued 1)

**Theorem 1.5. Euler Product Representation.**
For any real $\sigma$ with $\sigma > 1$, $\zeta(\sigma) = \prod_p (1 - p^{-\sigma})^{-1}$.

**Proof (continued).** Next consider,

$$(1 - 2^{-\sigma})(1 - 3^{-\sigma})\zeta(\sigma) = (1 - 3^{-\sigma}) \left( 1 + \sum_{p \mid n \Rightarrow p > 2} \frac{1}{n^\sigma} \right)$$

$$= 1 + \sum_{p \mid n \Rightarrow p > 2} \frac{1}{n^\sigma} - 3^{-\sigma} - \sum_{p \mid n \Rightarrow p > 2} \frac{1}{(3n)^\sigma}$$

$$= 1 + \sum_{p \mid n, n \neq 3, \Rightarrow p > 2} \frac{1}{n^\sigma} - \sum_{p \mid n \Rightarrow p > 2} \frac{1}{(3n)^\sigma} = 1 + \sum_{p \mid n \Rightarrow p > 3} \frac{1}{n^\sigma},$$

where in the final term we are summing over all $n$ such that if prime $p$ divides $n$, then $p > 3$.

# Theorem 1.5 (continued 2)

**Proof (continued).** Inductively, we then have for primes 2, 3, 5, . . . , $P$ that

$$(1-2^{-\sigma})(1-3^{-\sigma})(1-5^{-\sigma})\cdots(1-P^{-\sigma}\zeta(\sigma)=1+\sum_{p\,|\,n\Rightarrow p>P}\frac{1}{n^\sigma},$$

where in the final term we are summing over all $n$ such that if prime $p$ divides $n$, then $p>P$. Notice that if for $n$ with only prime divisors $p$ where $p>P$, we must have $n>P$. Hence, $\displaystyle\sum_{n=P}^{\infty}\frac{1}{n^\sigma}\geq\sum_{p\,|\,n\Rightarrow p>P}\frac{1}{n^\sigma}$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^\sigma}$ converges (it is a $p$-series with $p=\sigma>1$), then for any given $\varepsilon>0$, we can make $P$ sufficiently large so that $\displaystyle\sum_{n=P}^{\infty}\frac{1}{n^\sigma}<\varepsilon$ (from the definition of convergent series).

# Theorem 1.5 (continued 2)

**Proof (continued).** Inductively, we then have for primes 2, 3, 5, ..., $P$ that

$$(1-2^{-\sigma})(1-3^{-\sigma})(1-5^{-\sigma})\cdots(1-P^{-\sigma}\zeta(\sigma) = 1 + \sum_{p \mid n \Rightarrow p > P} \frac{1}{n^\sigma},$$

where in the final term we are summing over all $n$ such that if prime $p$ divides $n$, then $p > P$. Notice that if for $n$ with only prime divisors $p$ where $p > P$, we must have $n > P$. Hence, $\displaystyle\sum_{n=P}^{\infty} \frac{1}{n^\sigma} \geq \sum_{p \mid n \Rightarrow p > P} \frac{1}{n^\sigma}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^\sigma}$ converges (it is a $p$-series with $p = \sigma > 1$), then for any given $\varepsilon > 0$, we can make $P$ sufficiently large so that $\displaystyle\sum_{n=P}^{\infty} \frac{1}{n^\sigma} < \varepsilon$ (from the definition of convergent series).

# Theorem 1.5 (continued 3)

**Theorem 1.5. Euler Product Representation.**
For any real $\sigma$ with $\sigma > 1$, $\zeta(\sigma) = \prod_p (1 - p^{-\sigma})^{-1}$.

**Proof (continued).** We now have

$$\lim_{P\to\infty} (1 - 2^{-\sigma})(1 - 3^{-\sigma}) \cdots (1 - P^{-\sigma})\zeta(\sigma) = \lim_{P\to\infty} \left( \sum_{n=P}^{\infty} \frac{1}{n^\sigma} \right),$$

or $\prod_p (1 - p^{-\sigma})\zeta(\sigma) = 1$, or

$$\zeta(\sigma) = \frac{1}{\prod_p (1 - p^{-\sigma})} = \prod_p (1 - p^{-\sigma})^{-1},$$

as claimed. $\square$

# Theorem 1.3

**Theorem 1.3.** The series $\displaystyle\sum_{p \in \mathbb{P}} \frac{1}{p}$ diverges.

**Proof.** We take the logarithm of the Euler product representation of the zeta function and use the fact that the logarithm is continuous so that we can pass limits in and out of the logarithm function (including infinite products). So

$$\log(\zeta(\sigma)) = \log\left(\prod_p (1-p^{-\sigma})^{-1}\right) = -\sum_p (1-p^{-\sigma}),$$

Now the Maclaurin series for $\log(1+x) = \displaystyle\sum_{m=1}^{\infty} (-1)^{m-1}\frac{x^m}{m}$ (valid for $-1 < x \leq 1$), so

$$\log(1-p^{-\sigma}) = \sum_{m=1}^{\infty} (-1)^{m-1}\frac{(-p^{-\sigma})^m}{m} = \sum_{m=1}^{\infty} \frac{-p^{-m\sigma}}{m} = \sum_{m=1}^{\infty} \frac{-1}{mp^{m\sigma}}.$$

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# Theorem 1.3 (continued 1)

**Proof (continued).** So

$$\log(\zeta(\sigma)) = -\sum_p \log(1-p^{-\sigma}) = -\sum_p \sum_{m=1}^{\infty} \frac{-1}{mp^{m\sigma}}$$

$$= \sum_p \frac{1}{p^\sigma} + \sum_p \sum_{m=2}^{\infty} \frac{1}{mp^{m\sigma}}, \qquad (1.10)$$

since the series converges absolutely and can be rearranged. For any prime $p$, $1 - 1/p^\sigma \geq 1/2$ or $2 \geq 1/(1-p^{-\sigma})$, so

$$\sum_p \sum_{m=2}^{\infty} \frac{1}{mp^{m\sigma}} < \sum_p \sum_{m=2}^{\infty} \frac{1}{p^{m\sigma}} \text{ replacing the } \frac{1}{m} \text{ terms with } 1$$

$$= \sum_p \frac{1/p^{2\sigma}}{1-p^{-\sigma}} \text{ summing the geometric series}$$

with first term $1/p^{2\sigma}$ with ratio $p^{-\sigma}$

# Theorem 1.3 (continued 2)

**Proof (continued).** ...

$$\begin{aligned}\sum_p \sum_{m=2}^{\infty} \frac{1}{mp^{m\sigma}} &< \sum_p \frac{1/p^{2\sigma}}{1-p^{-\sigma}} \\ &\leq 2\sum_p \frac{1}{p^{2\sigma}} \text{ since } 1/(1-p^{-\sigma}) \leq 2 \\ &\leq 2\sum_{n=1}^{\infty} \frac{1}{n^{2\sigma}} = 2\zeta(2\sigma) < 2\zeta(2),\end{aligned}$$

since $\zeta(s)$ is a strictly decreasing function of $s$ for $s > 1$. The double sum $\displaystyle\sum_p \sum_{m=2}^{\infty} \frac{1}{mp^{m\sigma}}$ is therefore bounded by $2\zeta(2)$ for $\sigma \geq 1$, and is bounded by 2 times the $p$-series $\displaystyle\sum_p \frac{1}{p^{2\sigma}}$ when $\sigma > 1/2$.

# Theorem 1.3 (continued 3)

**Theorem 1.3.** The series $\displaystyle\sum_{p\in\mathbb{P}} \frac{1}{p}$ diverges.

**Proof (continued).** That is, $\displaystyle\sum_p \sum_{m=2}^{\infty} \frac{1}{mp^{m\sigma}} = O(1)$ (see the Introduction for the information on rates of growth). Therefore, by equation (1.10)

$$\log \zeta(\sigma) = \sum_p \frac{1}{p^\sigma} + O(1).$$

Now as $\sigma \to 1^+$ we have $\zeta(\sigma) \to \infty$ (see Figure 1.2, for example), so we must have $\displaystyle\lim_{\sigma\to 1^+} \sum_p \frac{1}{p^\sigma} = \infty$. That is, $\displaystyle\sum_p \frac{1}{p}$ diverges to infinity, as claimed. $\square$