

The Prime Number Theorem

Section 1.2. Arithmetic Functions—Proofs of Theorems



()

The Prime Number Theorem

April 9, 2022 1 / 6

Proposition 1.2.1

Proposition 1.2.1

Proposition 1.2.1. Suppose that $n > 1$, with prime factorization $n = \prod_{j=1}^m p_j^{k_j}$. Then

$$\tau(n) = \prod_{j=1}^m (k_j + 1), \quad \omega(n) = m, \quad \Omega(n) = \sum_{j=1}^m k_j.$$

Proof. The expressions for $\omega(n)$ and $\Omega(n)$ are just symbolic representations the definition of each. Divisors of $n = \prod_{j=1}^m p_j^{k_j}$ are of the

form $\prod_{j=1}^m p_j^{r_j}$ where for each j , $0 \leq r_j \leq k_j$. To count the factors, we notice that there are $k_j + 1$ possible values for k_j (namely, $0, 1, \dots, k_j$) and this holds for each $1 \leq j \leq m$.

()

The Prime Number Theorem

April 9, 2022 3 / 6

Proposition 1.2.1

Proposition 1.2.1 (continued)

Proposition 1.2.1. Suppose that $n > 1$, with prime factorization $n = \prod_{j=1}^m p_j^{k_j}$. Then

$$\tau(n) = \prod_{j=1}^m (k_j + 1), \quad \omega(n) = m, \quad \Omega(n) = \sum_{j=1}^m k_j.$$

Proof (continued). So there are $(k_1 + 1)$ choices for k_1 , $(k_2 + 1)$ choices for k_2 , \dots , $(k_m + 1)$ choices for k_m . By The Fundamental Counting Principle (see [Section 2.2. Counting Methods](#) in my online notes for Foundations of Probability and Statistics—Calculus Based [MATH 2250] or [Section 1.1. The Fundamental Counting Principle](#) in my online notes for Applied Combinatorics and Problem Solving [MATH 3340]) this means that there are $\prod_{j=1}^m (k_j + 1)$ possible different divisors of n , as claimed. \square

()

The Prime Number Theorem

April 9, 2022 4 / 6

Proposition 1.2.2

Proposition 1.2.2

Proposition 1.2.2. Write $S_\tau(x) = \sum_{n \leq x} \tau(n)$ and $S_\omega(x) = \sum_{n \leq x} \omega(n)$. Then

$$S_\tau(x) = \sum_{j \leq x} \left\lfloor \frac{x}{j} \right\rfloor \quad \text{and} \quad S_\omega(x) = \sum_{p \in P[x]} \left\lfloor \frac{x}{p} \right\rfloor.$$

Proof. Since $\tau(n)$ is the number of divisors of n , then we have

$$\tau(n) = |\{j \mid 1 \leq j \leq n, j \mid n\}| = |\{(j, n) \mid 1 \leq j \leq n, j \mid n\}|.$$

Let $x \in \mathbb{R}$ (where we may as well take $x \geq 1$). Fix j where $1 \leq j \leq x$. Then $|\{(j, n) \mid j \mid n\}| = |\{(j, n) \mid n = rj\}| = |\{r \mid n = rj\}| = |\{r \mid rj \leq x\}|$. The number of multiples of j that are less than or equal to x is $\lfloor x/j \rfloor$. That is, $|\{r \mid rj \leq x\}| = \lfloor x/j \rfloor$. Summing over all $1 \leq j \leq x$, we get the total number of divisors of positive integers less than or equal to x :

$$S_\tau(x) = \sum_{n \leq x} \tau(n) = \sum_{j \leq x} \left\lfloor \frac{x}{j} \right\rfloor, \text{ as claimed.}$$

()

The Prime Number Theorem

April 9, 2022 5 / 6

Proposition 1.2.2 (continued)

Proposition 1.2.2. Write $S_\tau(x) = \sum_{n \leq x} \tau(n)$ and $S_\omega(x) = \sum_{n \leq x} \omega(n)$.
Then

$$S_\tau(x) = \sum_{j \leq x} \left\lfloor \frac{x}{j} \right\rfloor \quad \text{and} \quad S_\omega(x) = \sum_{p \in P[x]} \left\lfloor \frac{x}{p} \right\rfloor.$$

Proof (continued). Since $\omega(n)$ is the number of prime divisors of n , then we have $\omega(n) = |\{p \mid p \text{ is prime, } p \mid n\}| = |\{(p, n) \mid p \text{ is prime, } p \mid n\}|$. Let $x \in \mathbb{R}$ (where we may as well take $x \geq 1$). Fix p prime where $1 \leq p \leq x$. Then $|\{(p, n) \mid p \mid n\}| = |\{(p, n) \mid p \text{ is prime, } n = rp\}| = |\{r \mid n = rp, p \text{ is prime}\}| = |\{r \mid rp \leq x, p \text{ is prime}\}|$. The number of multiples of p that are less than or equal to x is $[x/p]$. That is, $|\{r \mid rp \leq x\}| = [x/p]$. Summing over all prime $p \leq x$, we get the total number of prime divisors of positive integers less than or equal to x :

$$S_\omega(x) = \sum_{n \leq x} \omega(n) = \sum_{p \in P[x]} \left\lfloor \frac{x}{p} \right\rfloor, \text{ as claimed.} \quad \square$$