The Prime Number Theorem

Section 1.2. Arithmetic Functions—Proofs of Theorems

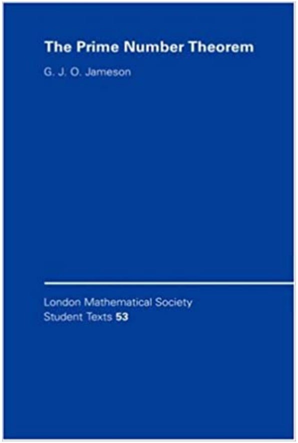

Table of contents

Proposition 1.2.1

Proposition 1.2.1. Suppose that $n > 1$, with prime factorization $n = \prod_{j=1}^{m} p_j^{k_j}$. Then

$$\tau(n) = \prod_{j=1}^{m}(k_j + 1), \;\; \omega(n) = m, \;\; \Omega(n) = \sum_{j=1}^{m} k_j.$$

Proof. The expressions for $\omega(n)$ and $\Omega(n)$ are just symbolic representations the definition of each. Divisors of $n = \prod_{j=1}^{m} p_j^{k_j}$ are of the form $\prod_{j=1}^{m} p_j^{r_j}$ where for each j, $0 \leq r_j \leq k_j$. To count the factors, we notice that there are $k_j + 1$ possible values for k_j (namely, $0, 1, \ldots, k_j$) and this holds for each $1 \leq j \leq m$.

Proposition 1.2.1

Proposition 1.2.1. Suppose that $n > 1$, with prime factorization $n = \prod_{j=1}^{m} p_j^{k_j}$. Then

$$\tau(n) = \prod_{j=1}^{m}(k_j + 1), \;\; \omega(n) = m, \;\; \Omega(n) = \sum_{j=1}^{m} k_j.$$

Proof. The expressions for $\omega(n)$ and $\Omega(n)$ are just symbolic representations the definition of each. Divisors of $n = \prod_{j=1}^{m} p_j^{k_j}$ are of the form $\prod_{j=1}^{m} p_j^{r_j}$ where for each j, $0 \leq r_j \leq k_j$. To count the factors, we notice that there are $k_j + 1$ possible values for k_j (namely, $0, 1, \ldots, k_j$) and this holds for each $1 \leq j \leq m$.

Proposition 1.2.1 (continued)

Proposition 1.2.1. Suppose that $n > 1$, with prime factorization $n = \prod_{j=1}^{m} p_j^{k_j}$. Then

$$\tau(n) = \prod_{j=1}^{m}(k_j + 1),\ \ \omega(n) = m,\ \ \Omega(n) = \sum_{j=1}^{m} k_j.$$

Proof (continued). So there are $(k_1 + 1)$ choices for k_1, $(k_2 + 1)$ choices for k_2, ..., $(k_m + 1)$ choices for k_m. By The Fundamental Counting Principle (see Section 2.2. Counting Methods in my online notes for Foundations of Probability and Statistics—Calculus Based [MATH 2250] or Section 1.1. The Fundamental Counting Principle in my online notes for Applied Combinatorics and Problem Solving [MATH 3340]) this means that there are $\prod_{j=1}^{m}(k_j + 1)$ possible different divisors of n, as claimed. □

Proposition 1.2.2

Proposition 1.2.2. Write $S_\tau(x) = \sum_{n\le x} \tau(n)$ and $S_\omega(x) = \sum_{n\le x} \omega(n)$. Then

$$S_\tau(x) = \sum_{j\le x} \left[\frac{x}{j}\right] \quad \text{and} \quad S_\omega(x) = \sum_{p\in P[x]} \left[\frac{x}{p}\right].$$

Proof. Since $\tau(n)$ is the number of divisors of n, then we have

$$\tau(n) = |\{j \mid 1 \le j \le n, j \mid n\}| = |\{(j,n) \mid 1 \le j \le n, j \mid n\}|.$$

Let $x \in \mathbb{R}$ (where we may as well take $x \ge 1$). Fix j where $1 \le j \le x$. Then $|\{(j,n) \mid j \mid n\}| = |\{(j,n) \mid n = rj\}| = |\{r \mid n = rj\}| = |\{r \mid rj \le x\}|$. The number of multiples of j that are less than or equal to x is $[x/j]$. That is, $|\{r \mid rj \le x\}| = [x/j]$. Summing over all $1 \le j \le x$, we get the total number of divisors of positive integers less than or equal to x: $S_\tau(x) = \sum_{n\le x} \tau(n) = \sum_{j\le x} \left[\frac{x}{j}\right]$, as claimed.

Proposition 1.2.2

Proposition 1.2.2. Write $S_\tau(x) = \sum_{n \le x} \tau(n)$ and $S_\omega(x) = \sum_{n \le x} \omega(n)$. Then

$$S_\tau(x) = \sum_{j \le x} \left[\frac{x}{j}\right] \quad \text{and} \quad S_\omega(x) = \sum_{p \in P[x]} \left[\frac{x}{p}\right].$$

Proof. Since $\tau(n)$ is the number of divisors of n, then we have

$$\tau(n) = |\{j \mid 1 \le j \le n, j \mid n\}| = |\{(j,n) \mid 1 \le j \le n, j \mid n\}|.$$

Let $x \in \mathbb{R}$ (where we may as well take $x \ge 1$). Fix j where $1 \le j \le x$. Then $|\{(j,n) \mid j \mid n\}| = |\{(j,n) \mid n = rj\}| = |\{r \ \mid n = rj\}| = |\{r \mid rj \le x\}|$. The number of multiples of j that are less than or equal to x is $[x/j]$. That is, $|\{r \mid rj \le x\}| = [x/j]$. Summing over all $1 \le j \le x$, we get the total number of divisors of positive integers less than or equal to x:
$S_\tau(x) = \sum_{n \le x} \tau(n) = \sum_{j \le x} \left[\frac{x}{j}\right]$, as claimed.

Proposition 1.2.2 (continued)

Proposition 1.2.2. Write $S_\tau(x) = \sum_{n \leq x} \tau(n)$ and $S_\omega(x) = \sum_{n \leq x} \omega(n)$. Then

$$S_\tau(x) = \sum_{j \leq x} \left[\frac{x}{j}\right] \quad \text{and} \quad S_\omega(x) = \sum_{p \in P[x]} \left[\frac{x}{p}\right].$$

Proof (continued). Since $\omega(n)$ is the number of prime divisors of n, then we have $\omega(n) = |\{p \mid p \text{ is prime}, p \mid n\}| = |\{(p, n) \mid p \text{ is prime}, p \mid n\}|$. Let $x \in \mathbb{R}$ (where we may as well take $x \geq 1$). Fix p prime where $1 \leq p \leq x$. Then $|\{(p, n) \mid p \mid n\}| = |\{(p, n) \mid p \text{ is prime}, n = rp\}| = |\{r \mid n = rp, p \text{ is prime}\}| = |\{r \mid rp \leq x, p \text{ is prime}\}|$. The number of multiples of p that are less than or equal to x is $[x/p]$. That is, $|\{r \mid rp \leq x\}| = [x/p]$. Summing over all prime $p \leq x$, we get the total number of prime divisors of positive integers less than or equal to x:

$$S_\omega(x) = \sum_{n \leq x} \omega(n) = \sum_{p \in P[x]} \left[\frac{x}{p}\right], \text{ as claimed.}$$

□