Chapter 1. A Brief History of Prime

Note. In this chapter, we review several classical results concerning prime numbers. We consider Euclid's proof that there are infinitely many primes, the sieve of Eratosthenes,the Euclidean Algorithm, Mersenne primes, Fermat's Little Theorem, Fermat numbers (i.e., Fermat primes), conditions for primality (often involving congruences). We also introduce some more modern ideas, like summing over primes (and bounds on these quantities) and the Riemann zeta function. We consider the Fundamental Theorem of Arithmetic in Section 1.6 (and again in Section 2.2, in Chapter 4, and throughout the book). Given its central role, we start with a statement of the Fundamental Theorem of Arithmetic.

Theorem 1.1. The Fundamental Theorem of Arithmetic.
Every integer greater than 1 can be expressed as a product of prime numbers in a way that is unique up to order.

Section 1.1. Euclid and Primes

Note. Euclid's famous proof that there are infinitely many primes appears in his *Elements of Geometry* as Proposition 20 in Book IX. Some discussion of the history can be found in my online notes for Elementary Number Theory (MATH 3120) on Section 2. Unique Factorization and my online notes for Introduction to Modern Geometry (MATH 4157/5157) on Section 2.4. Books VII and IX. Number Theory. In this section we consider two proofs, Euclid's original proof and an analytic proof of Euler.

Theorem 1.2. Euclid's Infinite Primes Theorem. There are infinitely many primes.

Note. We now consider a second proof of Theorem 1.2 by Euler. Euler's proof is analytic in the sense that he appeals to the divergence of the harmonic series to establish the infinity claim.

Note. Everest and Ward give a straightforward argument that the harmonic series diverges, because

$$\frac{1}{2^k+1} + \frac{1}{2^k+2} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}} \geq 2^k \cdot \frac{1}{2^{k+1}} = \frac{1}{2},$$

that is, summing reciprocals of the integers between $2^k + 1$ and 2^{k+1} (inclusive) results in a quantity that is at least 1/2. So we have the sum $\sum_{n=1}^{2^{k+1}} \frac{1}{n} \geq \frac{k}{2}$ for all $k \geq 1$, hence the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. This technique suggests that the rate at which the harmonic series diverges is very slow, since we only are insured that we have a partial sum of size $k/2$ when we have summed 2^{k+1} terms. So if we want the partial sum to be at least, say, 20 then we need to sum $2^{21} = 2{,}097{,}152$ terms; if we want the partial sums to be at least 100 then we need to sum $2^{101} \approx 2.535 \times 10^{30}$ terms!

Note 1.1.A. In Calculus 2 (MATH 1920) you probably showed that the harmonic series diverges using the Integral Test; see my online Calculus 2 notes on Section 10.3. The Integral Test where there is information on the slow rate of divergence of the harmonic series base on the logarithm function. Everest and Ward illustrate the argument in Figure 1.1 from which we have the inequality

$$\log(N+1) = \int_0^N \frac{1}{x+1}\,dx \le \sum_{n=1}^{N} \frac{1}{n} \le 1 + \int_1^N \frac{1}{x}\,dx = 1 + \log N$$

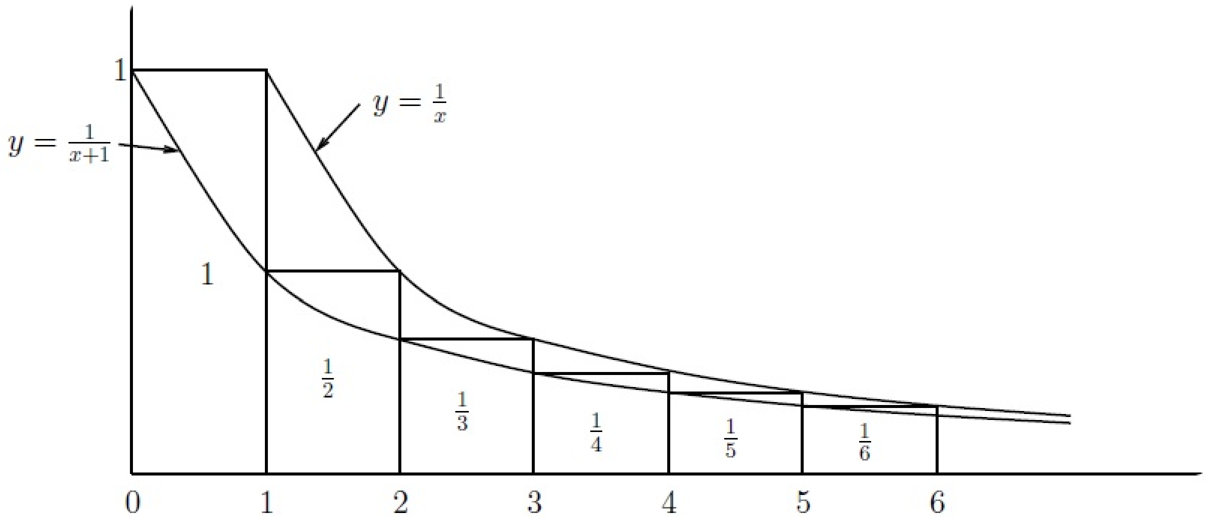

Figure 1.1. Graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x+1}$ trapping the harmonic series.