Section 1.2. Summing over the Primes

Note. In this section we consider the a series similar to the harmonic series, but which involves summing over the primes $\mathbb{P}$ instead of summing over the natural numbers $\mathbb{N}$. We give three proofs showing that the series diverges. We also define the Riemann zeta function and give the Euler product representation of it

Theorem 1.3. The series $\displaystyle\sum_{p\in\mathbb{P}} \frac{1}{p}$ diverges.

Note 1.2.A. We now consider a second proof of Theorem 1.3. In the second proof, we put a lower bound on a partial sum of the series and show that this lower bound approaches infinity:

$$\sum_{p\leq N} \frac{1}{p} > \log\log N - 2. \tag{14}$$

Here, and in the rest of this section, we let p denote a prime number so that "$p \leq N$" is shorthand for "$p \in \mathbb{P}, p \leq N$."

Note. In 1874, Franz Mertens proved three results concerning bounds on the series of Theorem 1.3. He improved the inequality in Note 1.2.A and proved:

$$\sum_{p\leq N} \frac{1}{p} = \log\log N + A + O\left(\frac{1}{\log N}\right),$$

Where A is a constant (approximately 0.261). His result appeared in "Ein Beitrag zur analytischen Zahlentheorie," *J. Reine Angew. Math.*, **78**, 46–62 (1874). This is available online (in German, of course) on the Göttingen Digitization Center (accessed 4/14/2022).

Note. In preparation for a third proof of Theorem 1.3, we introduce the Riemann zeta function in the form given by Everest and Ward.

Definition 1.4. The *Riemann zeta function* is defined as $\zeta(\sigma) = \sum_{n=1}^{\infty} \frac{1}{n^\sigma}$, "wherever this makes sense."

Note. This disclaimer of "wherever this makes sense" is not optimal! We know that if σ is real and $\sigma > 1$, then $\zeta(\sigma)$ equals the sum of the p-series where $p = \sigma$ (we see that a p-series converges for all $p > 1$ converges in Calculus 2 [MATH 1920] by the Integral Test; see my online Calculus 2 notes on Section 10.3. The Integral Test for details)...though we know the actual value of $\zeta(\sigma)$ for very few σ (for example, $\zeta(2) = \pi^2/6$). For our purposes in this section, we can take σ and real and greater than 1. It turns out that Definition 1.4 is also valid for every complex σ with $\mathrm{Re}(\sigma) > 1$. In fact, the Riemann zeta function is defined in all of the complex plane, expect at 1 (though the formula given in Definition 1.4 is only valid for $\mathrm{Re}(\sigma) > 1$). Not surprisingly, a detailed definition of the Riemann zeta function requires knowledge of complex analysis. Details of the definition valid in $\mathbb{C} \setminus \{1\}$ are given in my online Complex Analysis 2 (MATH 5520) notes on Section VII.8. The Riemann Zeta Function (though the Complex Analysis 2 class never gets this far into the material). This function is related to the Prime Number Theorem (which concerns an estimation of the number of primes less than or equal to a given value); for an explanation, see my online notes on Supplement. The Prime Number Theorem—History. The location of the zeros of the Riemann zeta function are also intimately related to the error term in the estimation given by the Prime Number Theorem. An explanation of this is given in Supplement. The Riemann Hypothesis—History. These supplements are intended for use in Elementary Number Theory (MATH 3120). Some of these ideas are also covered in this class in Chapters 8 and 9. The (uninteresting) graph of $\zeta(\sigma)$ for σ real and $\sigma > 1$ is given in Figure 1.2. We see more dynamic behavior if we consider a "slice" of the real part of $\zeta(\sigma)$ for $\mathrm{Re}(\sigma) = 3/2$.

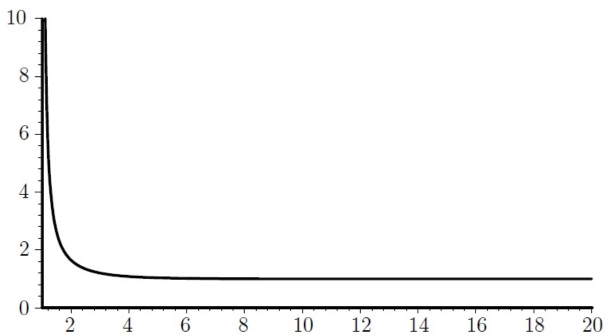

Figure 1.2. The graph of $\zeta(\sigma)$ for $1 < \sigma \leqslant 20$.

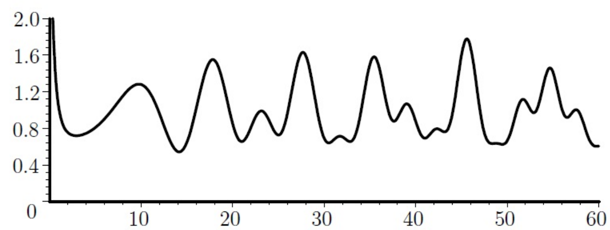

Figure 1.3. The graph of $\Re(\zeta(\frac{3}{2} + it))$ for $0 \leqslant t \leqslant 60$.

Note. We now state and prove Euler's Product Representation of the zeta function. Our proof (like Euler's original) is restricted to σ real and $\sigma > 1$. However, the representation holds for all $\mathrm{Re}(\sigma) > 1$; see Theorem VII.8.17 in my Complex Analysis 2 notes Section VII.8. The Riemann Zeta Function .

Theorem 1.5. Euler Product Representation.

For any real σ with $\sigma > 1$, $\zeta(\sigma) = \displaystyle\prod_p (1 - p^{-\sigma})^{-1}$.

Note. An infinite product is defined in terms of partial products, similar to infinite sums (i.e., series). We will take logarithms of these products, so this imposes certain nonzero constraints. We are now ready to present a third proof of Theorem 1.3, this time using the Euler Product Representation.

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