Chapter 1. Foundations

Note. The goal of this course is, for $\pi(x)$ and the number of prime numbers less than or equal to x, to find an (asymptotic) approximation of $\pi(x)$. We vaguely use the term "arithmetic function" to indicate a sequence defined using some number theoretic properties. For example, with u_P as the arithmetic function

$$u_P(n) = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise,} \end{cases}$$

we can express $\pi(x)$ and $\displaystyle\sum_{n \le x} u_P(x)$. In this chapter, we will consider Abel summation and integral estimation to approximate such a sum.

Note. A somewhat detailed history of prime numbers and the Prime Number Theorem can be found in my online notes for Elementary Number Theory (MATH 3120) on Supplement. The Prime Number Theorem—History, so we only mention here a brief outline of the history. In the early 1850s, Russian mathematician Pafnuty Chebyshev (May 16, 1821–December 8, 1894) introduced the function $\theta(x) = \sum_{p \in P[x]} \log p$, where $P[x]$ denotes the set of primes not greater than x) and proved that $\displaystyle\lim_{x\to\infty} \frac{\theta(x)}{x} = 1$ is equivalent to the Prime Number Theorem. He was able to show that $\pi(x)$ is *close* to $x/\log x$ in the sense that

$$0.92129 \le \liminf_{x\to\infty} \frac{\pi(x)}{x/\log x} \le 1 \le \limsup_{x\to\infty} \frac{\pi(x)}{x/\log x} \le 1.10555.$$

See page 606 of J. L. Goldstein's "A History of the Prime Number Theorem," *The American Mathematical Monthly*, **80**(6), 599–615 (1973); this paper is available through JSTOR (accessed 3/31/2022).

Note/Definition. For an arithmetic function $a(n)$, we will have a *Dirichlet series* as a series of the form $\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$, where s is a complex variable. For example, with $a(n) = 1$ this defines the *Riemann zeta function* (for $\text{Re}(s) > 1$). We will use (in Chapter 3) the nature of a Dirichlet series to get information about the partial sums of arithmetic function $a(n)$.

Section 1.1. Counting Prime Numbers

Note. In this section we give a brief history of the Prime Number Theorem, up through the "elementary proof" of Selberg and Erődos in 1949. We consider three functions as candidate asymptotic estimates of $\pi(x)$ and claim that, by our method of estimation, each is equivalent to the other as an estimate.

Note. Our first result concerning prime numbers is due to Euclid, and appears in his *Elements of Geometry* as Proposition 20 in Book IX where it is stated as "Prime numbers are more than any assigned multitude of prime numbers." Euclid's proof is online in David Joyce's online version of Euclid's Elements (accessed 3/30/2022). We now state and prove the result (though we require some results from Elementary Number Theory [MATH 3120]).

Proposition 1.1.1. There are infinitely many prime numbers.

Note. In the proof of Proposition 1.1.1, we see that there must be some prime divisor of $p_1 p_2 \cdots p_n + 1$ different from $p_1, p_2, \ldots, p_n$ (in increasing order). So there must be some prime p_{n+1} strictly between p_n and $p_1 p_2 \cdots p_n + 1$. In a related result, we have:

Exercise 1.1.3. Let the primes be listed in order as $p_1, p_2, \ldots$. Then $p_n < 2^{2^{n-1}}$ for each $n \in \mathbb{N}$. From this we have $\pi(x) \geq \dfrac{\log \log x}{\log 2}$.

We now turn our attention to a more precise approximation of $\pi(x)$.

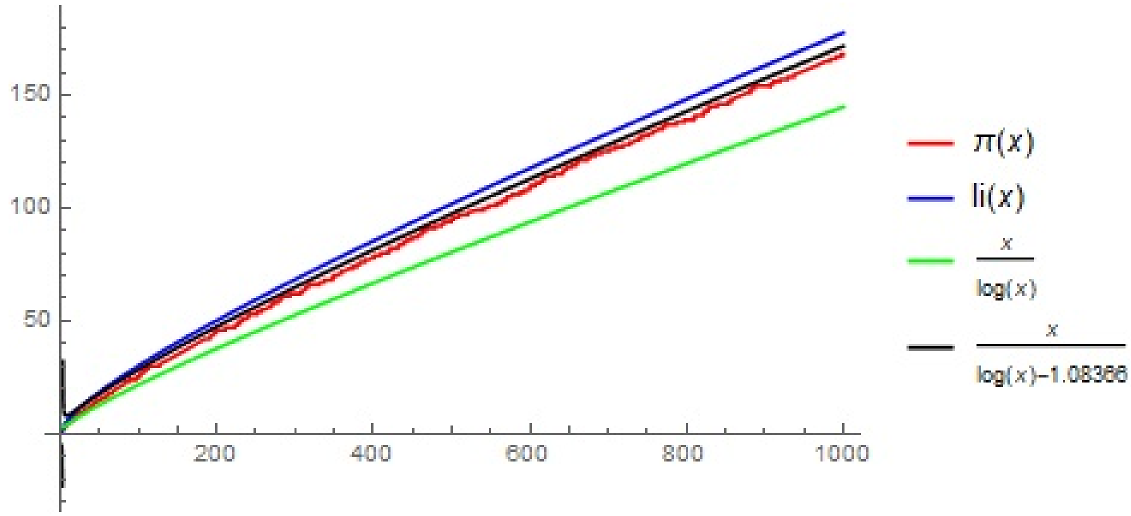
Note. Based on tables of primes, three functions were proposed as estimates of $\pi(x)$ (again, see my online notes for Elementary Number Theory (MATH 3120) on Supplement. The Prime Number Theorem—History for more historical details). We consider $x/\log x$, $x/(\log x - 1)$, and $\mathrm{li}(x)$ where $\mathrm{li}(x)$ is the logarithmic integral (also denoted $\mathrm{Li}(x)$):

$$\mathrm{li}(x) = \int_2^x \frac{1}{\log t}\, dt.$$

As circumstantial evidence for these estimates, consider the following table from page 3:

n	$\pi(n)$	$n/\log n$	$n/(\log n - 1)$	$\mathrm{li}(n)$
1,000	168	145	169	177
10,000	1,229	1,086	1,218	1,246
50,000	5,133	4,621	5,092	5,166
100,000	9,592	8,686	9,512	9,630
500,000	41,538	38,103	41,246	41,607
1,000,000	78,498	72,382	78,031	78,628
10,000,000	664,579	620,421	661,459	664,918

A graph of $\pi(x)$, alongside a graphs of $x/\log x$, $\mathrm{li}(x)$, and $x/(\log x - 1.08366)$ is in the following image from an MAA website on "The Origin of the Prime Number Theorem: A Primary Source Project for Number Theory Students:



Notice that, over this range of input values, $x/\log x$ are the weakest of the approximating functions.

Definition. Given two functions f and g defined for positive real numbers, we denote the limit

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = 1$$

as $f(x) \sum g(x)$ as $x \to \infty$.

Note. In Section 1.5. The Function $li(x)$, we will see that

$$\frac{x}{\log x} \sim \frac{x}{\log x - 1} \sim \mathrm{li}(x) \text{ as } x \to \infty.$$

Since $\sim$ is an equivalence relation (see Exercise 1.1.A), then we can take as the Prime Number Theorem the condition $\pi(x) \sim f(x)$ for $f(x)$ equal to any of $\dfrac{x}{\log x}$, $\dfrac{x}{\log x - 1}$, or $\mathrm{li}(x)$.

Note. The first step in the direction of analytic number theory was taken by Leonhard Euler (April 15, 1707–September 18, 1783) in 1737 when he proved (with some lack of rigor, by modern standards) that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \text{ for } s > 1.$$

Around 1800, Adrien-Marie Legendre (September 18, 1752–January 9, 1833) and Carl Friedrich Gauss (April 30, 1777–February 23, 1855) studied tables of primes and conjectured about the asymptotic behavior of $\pi(x)$. Legendre proposed that $\pi(x) \sim x/(\log x - 1.08366)$ and Gauss proposed (but never published) that $\pi \sim \mathrm{li}(x)$. In the early 1850s, Pafnuty Chebyshev (May 16, 1821–December 8, 1894) presented his bound on $\dfrac{\pi(x)}{x/\log x}$, as stated in the introduction to this chapter. The Bernhard Riemann enters.

Note. Bernhard Riemann (September 17, 1826–July 20, 1866) in a 9-page article "On the Number of Primes Less Than a Given Magnitude" (published in the November 1859 issue of *Monatsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*) formally introduced the zeta function as a function of a complex variable. This set the stage for the proof of the Prime Number Theorem (and laid the foundations of research that continues today). A translation appears in the appendix of Harold Edwards' *Riemann's Zeta Function*, Academic Press 1974 (reprinted by Dover Publications in 2001), and a translation is online on the Claymath.org website (accessed 3/6/2022). Riemann's definition of $\zeta(s)$ for

$\mathrm{Re}(z) > 1$ is the same as Euler's for $s > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \text{ for } \mathrm{Re}(s) > 1.$$

Riemann extended $\zeta(s)$ to the rest complex plane, except $s = 1$. The zeta function is the meromorphic on $\mathbb{C}$ with a simple pole at $s = 1$ only. This is established in Complex Analysis 2 (MATH 5520) in VII.8. The Riemann Zeta Function (though the class traditionally does not reach this point). The location of the "nontrivial" zeros of $\zeta(s)$ are relevant to the Prime Number Theorem. If the zeros are located in real part less than one, then the Prime Number Theorem will follow; see page 156 of John Derbyshire's *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*, Washington, DC: Joseph Henry Press (2003) "Riemann's 1859 paper gave an exact expression for the error term. That expression…involves all the non-trivial zeros of the zeta function, so the key to understanding the error term is hidden in among the zeros somehow." This quote is from page 234 of Derbyshire. Riemann did not show that the nontrivial zeros have real part less than one, and so he was unable to prove the Prime Number Theorem.

Note. In 1896, Jacques Hadamard (December 8, 1865–October 17, 1963) and Charles de la Vallée Poussin (August 14, 1866–March 2, 1962) independently proved the Prime Number Theorem by showing that the nontrivial zeros of the zeta function have real part less than one. So the proof is heavily dependent on the theory of functions of a complex variable. It took another 50 years for a proof to be given that did not depend on complex function theory. In 1949, Alte Selberg (June 14, 1917–August 6, 2007) and Paul Erdős (March 26, 1913–September 20, 1996) gave an "elementary" proof of the Prime Number Theorem; that is, they gave a proof that did not use complex function theory. This is the standard use of the terminology "elementary" in this setting; it is not meant to imply that the proof is simple!

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