Section 1.2. Arithmetic Functions

Note. In this section we give several examples of arithmetic sequences and consider some of their properties (including multiplicativity).

Note. Recall that a sequence of real numbers is simply a real valued function with domain $\mathbb{N}$ (as defined in Calculus 2, MATH 1920, in Section 10.1 Sequences). We take the same approach here, but we may have sequence values as either real or complex. For sequence a, we more commonly use the notation $a(n)$ instead of a_n as we did in calculus.

Example 1.2.A. Some examples of sequences are:

$$e_j(n) = \begin{cases} 1 & \text{if } n = j \\ 0 & \text{if } n \neq j. \end{cases}$$

Notice the resemblance of $e_j(n)$ to a unit vector. Given $E \subseteq \mathbb{N}$, define

$$u_E(n) = \begin{cases} 1 & \text{if } n \in E \\ 0 & \text{if } n \notin E. \end{cases}$$

In measure theory, u_E is called the *characteristic function* on set E (see my online notes on Real Analysis 1 on Section 3.2. Sequential Pointwise Limits). Some examples more focused on number theory are:

$$\begin{aligned} \tau(n) &= \text{the number of (positive) divisors of } n\text{, including 1 and } n \\ \omega(n) &= \text{the number of prime divisors of } n \\ \Omega(n) &= \text{the number of prime factors of } n\text{, counted with repetitions.} \end{aligned}$$

Some properties of these functions are given in the following proposition.

Proposition 1.2.1. Suppose that $n > 1$, with prime factorization $n = \prod_{j=1}^{m} p_j^{k_j}$. Then

$$\tau(n) = \prod_{j=1}^{m}(k_j + 1), \ \ \omega(n) = m, \ \ \Omega(n) = \sum_{j=1}^{m} k_j.$$

Definition. For a an arithmetic function, define the *summation function*

$$A(x) = \sum_{n \le x} a(n).$$

Note. If arithmetic function a counts something, then the summation function A gives a cumulative count. Notice that the function $\pi(x)$ introduced in the previous section is a summation function since $\pi(x) = \sum_{n \le x} u_P(n)$, where u_P is defined in the previous section:

$$u_P(n) = \begin{cases} 1 & 1 \text{ if } n \text{ is prime} \\ 0 & \text{otherwise.} \end{cases}$$

Note. We use the notation $[x]$ to indicate the greatest integer function; that is, $[x]$ is the greatest integer not greater than x. We let $P[x]$ indicate the set of primes not greater than x. Notice that $\pi(x) = \#P[x] = |P[x]|$ (where $\#$ and $|\cdot|$ indicate the cardinality of the given set).

Proposition 1.2.2. Write $S_\tau(x) = \sum_{n \le x} \tau(n)$ and $S_\omega(x) = \sum_{n \le x} \omega(n)$. Then

$$S_\tau(x) = \sum_{j \le x} \left[\frac{x}{j}\right] \quad \text{and} \quad S_\omega(x) = \sum_{p \in P[x]} \left[\frac{x}{p}\right].$$

Definition. Denote the greatest common divisor of integers m and n as (m, n). If $(m, n) = 1$ then m and n are *relatively prime* (or "*coprime*). An arithmetic function a is

completely multiplicative if $(a(mn) = a(m)a(n)$ for all m and n, and

multiplicative if $a(mn) = a(m)a(n)$ whenever $(m, n) = 1$.

Note. We consider multiplicative and completely multiplicative because we can determine the value of such a function a by knowing its values on powers of primes or on primes themselves, respectively. If $n = \prod_{j=1}^{r} p_j^{k_j}$ then for multiplicative a we have $a(n) = \prod_{j=1}^{r} a(p_j^{k_j})$; for completely multiplicative a we have $a(n) = \prod_{j=1}^{r} a(p_j)^{k_j}$.

Note. Some examples of multiplicative and completely multiplicative arithmetic functions are:

(i) For any s, let $a(n) = n^s$. Then a is completely multiplicative.

(iii) τ is multiplicative since for $(m, n) = 1$ we must have that m and n are powers of different primes and the result follows from Proposition 1.2.1. It is not complete multiplicative since $\tau(2) = 2$ and $\tau(4) = 3 \neq \tau(2)\tau(2)$.

(v) Neither ω and Ω is multiplicative. But $\Omega(mn) = \Omega(m) + \Omega(n)$, and for $(m, n) = 1$ we have $\omega(mn) = \omega(m) + \omega(n)$.

(vi) *Liouville's function* is defined as $\lambda(n) = (-1)^{\Omega(n)}$. By statement (v), it is completely multiplicative.