Section 1.3. Abel Summation

Note. In this section we

Note.

Proposition 1.3.1. For integers $n > m \leq 0$,

$$\sum_{r=m+1}^{n} a(r)f(r) = \sum_{r=m}^{n-1} A(r)[f(r) - f(r+1)] + A(n)f(n) - A(m)f(m).$$

In particular,

$$\sum_{r=1}^{n} a(r)f(r) = \sum_{r=1}^{n-1} A(r)[f(r) - f(r+1)] + A(n)f(n).$$

Proposition 1.3.2. Suppose that $f(r)$ is real and non-negative, and decreases with r. Suppose that $a(r)$ and $b(r)$ are such that $A(r) \leq CB(r)$ for all r. Then

$$\sum_{r=1}^{n} a(r)f(r) \leq C\sum_{r=1}^{n} b(r)f(r).$$

Proposition 1.3.3. Suppose that $A(n)f(n) \to 0$ as $n \to \infty$. Then if one of the series

$$\sum_{r=1}^{\infty} a(r)f(r) \quad \text{and} \quad \sum_{r=1}^{\infty} A(r)[f(r) - f(r+1)]$$

converges, then so does the other, to the same sum.

Proposition 1.3.4. Dirichlet's Test for Convergence.

Suppose that:

(i) $|A(r)| \leq C$ for all r,

(ii) $f(r_{\rightarrow}0$ as $r \to \infty$,

(iii) $\sum_{r=1}^{\infty} |f(r) - f(r_1)|$ is convergent.

Then $\sum_{r=1}^{\infty} a(r)f(r)$ converges, say to S, where $|S| \leq C\displaystyle\sum_{r=1}^{\infty}|f(r) - f(r+1)|$. Condition (iii) can be replaced by: (iiia) $f(r)$ is non-negative and decreasing. We then have $|S| \leq Cf(1)$.

Theorem 1.3.5. Let $y < x$, and let f be a function (with real or complex values) having a continuous derivative on $[y, x]$. Then

$$\sum_{y<r\leq x} a(r)f(r) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)\,dt.$$

Proposition 1.3.6. Let f hae a continuous derivative on $[1, x]$. Then:

(i) $\displaystyle\sum_{r\leq x} a(r)f(r) = A(x)f(x) - \int_1^x A(t)f'(t)\,dt,$

(ii) $\displaystyle\sum_{r\leq x} a(r)[f(x) - f(r)] = \int_1^x A(t)f'(t)\,dt.$

Proposition 1.3.7. If f has a continuous derivative on $[2, x]$ and $a(1) = 0$, then

$$\sum_{2\leq r\leq x} a(r)f(r) = A(x)f(x) = \int_2^x A(t)f'(t)\,dt.$$

Proposition 1.3.8. Suppose that f has a continuous derivative on $[1, \infty)$, and that $A(x)f(x) \to 0$ as $x \to \infty$. Then

$$\sum_{r=1}^{\infty} a(r)f(r) = -\int_1^{\infty} A(t)f'(t)\,dt,$$

in the sense that if either side converges, then so does the other, to the same value. Further, we then have

$$\sum_{r>x} a(r)f(r) = -A(x)f(x) - \int_x^{\infty} A(t)f'(t)\,dt.$$

Note.

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