Theory of Matrices, MATH 5090, Summer 2018

Homework 7, Section 3.3

Due Friday, June 29 at 1:00

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses.

3.11. The *affine group*, denoted $AL(n)$, consists of "doubletons" (A, v) where A is an $n \times n$ full rank matrix (so each A is invertible by Note 3.3.A) and $v \in \mathbb{R}^n$, we define the mapping $(A, v) : \mathbb{R}^n \to \mathbb{R}^n$ as $(A, v)x = Ax + v$. The binary operation $*$ on $AL(n)$ is defined as

$$((A, v) * (B, w))\, x = (A, v)\, ((B, w)x) = (A, v)(Bx + w)$$

$$= A(Bx + w) + v = ABx + Aw + v = (AB, Aw + v)$$

(so the binary operation is transformation composition; since function composition is associative, then the binary operation on $AL(n)$ is associative). Also, since A and B are full rank then both are invertible and so AB is full rank since $(AB)^{-1} = B^{-1}A^{-1}$.

(a) What is the identity in $AL(n)$?

(b) Let $(A, v) \in AL(n)$. What is the inverse of (A, v)?

Note. Since $AL(n)$ is closed under the binary operation, the binary operation is associative, there is an identity in $AL(n)$, and there are inverses in $AL(n)$ then $AL(n)$ actually is a group.

3.12. Prove Theorem 3.3.17. Let A and B be $n \times n$ full rank matrices.

(1) $A(I + A)^{-1} = (I + A^{-1})^{-1}$,

(2) $(A + BB^T)^{-1}B = A^{-1}B(I + B^TA^{-1}B)^{-1}$,

(6) $(I + AB)^{-1} = I - A(I + BA)^{-1}B$,

where we require the invertibility of relevant sums and differences.

3.3.A. Denote the set of all $n \times n$ full rank matrices (that is, invertible matrices; see Note 3.3.A) as $GL_n(\mathbb{R})$.

(a) Prove that $GL_n(\mathbb{R})$ is a *group* (the "general linear group") under matrix multiplication. Notice that $GL_n(\mathbb{R})$ is not an "Abelian group" (that is, matrix multiplication is not commutative).

3.3.B. Use the definition of matrix product to prove that for any $n \times n$ matrix H in Hermite form we have $H^2 = H$. NOTE: A square matrix A satisfying $A^2 = A$ is *idempotent*. HINT: Show that the ith row of H and the ith row of H^2 are the same. Consider two cases: Case 1 where $h_{ii} = 0$ (this is the easy case) and Case 2 where $h_{ii} = 1$. Justify your argument by quoting the three conditions given in the definition of "Hermite form."