Theory of Matrices, MATH 5090, Summer 2020

Homework 6, Sections 3.3 and 3.4, **CORRECTED**

Due Wednesday, July 1

Write in complete sentences!!! *Explain* what you are doing and convince me that you understand what you are doing and why. Justify all steps by quoting relevant results from the textbook or hypotheses.

3.12. Prove Theorem 3.3.17. Let A and B be $n \times n$ full rank matrices.

(1) $A(I + A)^{-1} = (I + A^{-1})^{-1}$,

(2) $(A + BB^T)^{-1}B = A^{-1}B(I + B^T A^{-1}B)^{-1}$,

(6) $(I + AB)^{-1} = I - A(I + BA)^{-1}B$,

where we require the invertibility of relevant sums and differences.

3.3.A. Denote the set of all $n \times n$ full rank matrices (that is, invertible matrices; see Note 3.3.A) as $GL_n(\mathbb{R})$.

(a) Prove that $GL_n(\mathbb{R})$ is a *group* (the "general linear group") under matrix multiplication. Notice that $GL_n(\mathbb{R})$ is not an "Abelian group" (that is, matrix multiplication is not commutative). HINT: Show the $GL_n(\mathbb{R})$ is closed under multiplication, there is an identity in $GL_n(\mathbb{R})$, we have associativity, and every element of $GL_n(\mathbb{R})$ has an inverse in $GL_n(\mathbb{R})$.

3.3.B. Use the definition of matrix product to prove that for any $n \times n$ matrix H in Hermite form we have $H^2 = H$. NOTE: A square matrix A satisfying $A^2 = A$ is *idempotent*. HINT: Show that the ith row of H and the ith row of H^2 are the same. Consider two cases: Case 1 where $h_{ii} = 0$ (this is the easy case) and Case 2 where $h_{ii} = 1$. Justify your argument by quoting the three conditions given in the definition of "Hermite form."

3.13. Prove Theorem 3.4.1: If A is a square nonsingular matrix and $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where both A_{11} and A_{22} are nonsingular then in terms of the Schur complement of A_{11} in A, $Z =$

$A_{22} - A_{21}A_{11}^{-1}A_{12}$, we have the inverse of A is

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}Z^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}Z^{-1} \\ -Z^{-1}A_{21}A_{11}^{-1} & Z^{-1} \end{bmatrix}.$$