Section 3.4. More on Partitioned Square Matrices: The Schur Complement

Note. In this section, we associate a quantity with a partitioned matrix and express the inverse (when it exists) and the determinant of a matrix in terms of this quantity.

Note. Recall that a full rank partitioning of a square $n \times n$ matrix $A$ of rank $r$ is $A = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$ where $W$ is an $r \times r$ full rank matrix, $X$ is $r \times (n - r)$, $Y$ is $(n - r) \times r$, and $Z$ is $(n - r) \times (n - r)$. So $[W X]$ is of full row rank $r$ and the rows of $[W X]$ span the row space of $A$. Also, $\begin{bmatrix} W \\ Y \end{bmatrix}$ is of full column rank $r$ and the columns of $\begin{bmatrix} W \\ Y \end{bmatrix}$ span the column space of $A$. So the rows of $[Y Z]$ can be written as linear combinations of the rows of $[W X]$ and so there is some $(n - r) \times r$ matrix $T$ such that $[Y Z] = T[W X]$. Similarly, there is $r \times (n - r)$ matrix $S$ such that $\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} W \\ Y \end{bmatrix} S$. So we have $Y = TW$, $Z = TX$, $X = WS$, and $Z = YS$, so that $Z = TX = TWS$. Since $W$ is of full rank, then $W^{-1}$ exists so that $T = YW^{-1}$, $S = W^{-1}X$, and $Z = YS = YW^{-1}X$ (or, equivalently, $Z = TX = YW^{-1}X$). So a full rank partitioning can be written in terms of $W$, $X$, and $Y$ only as

$$A = \begin{bmatrix} W & X \\ Y & YW^{-1}X \end{bmatrix}. \quad (*)$$
3.4. The Schur Complement

**Definition.** If $A$ is a square matrix partitioned as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where $A_{11}$ is nonsingular, then $Z = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is the Schur complement of $A_{11}$ in $A$.

**Note.** If $A_{11}$ is of full rank and $\text{rank}(A_{11}) = \text{rank}(A)$, then from $(\ast)$ we see that $A_{22} = A_{21}A_{11}^{-1}A_{12}$ and so in this case $Z = 0$.

**Note.** As described in the note after Theorem 3.3.6, for any $n \times m$ matrix $A$ of rank $r > 0$, there is a $n \times n$ permutation matrix $E_{\pi_1}$ and a $m \times m$ permutation matrix $E_{\pi_2}$ such that $E_{\pi_1}AE_{\pi_2}$ can be partitioned as $E_{\pi_1}AE_{\pi_2} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ where $B_{11}$ is a $r \times r$ full rank matrix. So from $(\ast)$ we have $E_{\pi_1}AE_{\pi_2} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{21}B_{11}^{-1}B_{12} \end{bmatrix}$.

We can then factor as:

$$
\begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix} = 
\begin{bmatrix}
B_{11} & \\
B_{21}
\end{bmatrix}
\begin{bmatrix}
I & B_{11}^{-1}B_{12} \\
B_{21}B_{11}^{-1}
\end{bmatrix}
\begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{21}B_{11}^{-1}B_{12}
\end{bmatrix}
$$

With $P = E_{\pi_1}^{-1}$ and $Q = E_{\pi_2}^{-1}$ we have

$$
A = P 
\begin{bmatrix}
B_{11} & \\
B_{21}
\end{bmatrix}
\begin{bmatrix}
I & B_{11}^{-1}B_{12} \\
B_{21}B_{11}^{-1}
\end{bmatrix}
Q = 
\begin{bmatrix}
P & \\
P B_{21}
\end{bmatrix}
\begin{bmatrix}
Q & B_{11}^{-1}B_{12}Q \\
QB_{21}B_{11}^{-1}
\end{bmatrix}
$$

and

$$
A = P 
\begin{bmatrix}
I & \\
B_{21}B_{11}^{-1}
\end{bmatrix}
\begin{bmatrix}
B_{11} & B_{12} \\
P B_{21}
\end{bmatrix}
Q = 
\begin{bmatrix}
P & \\
P B_{21}B_{11}^{-1}
\end{bmatrix}
\begin{bmatrix}
B_{11}Q & B_{12}Q \\
B_{11}QB_{21}B_{11}^{-1}
\end{bmatrix}
$$
Now $P$ and $Q$ are permutation matrices and so are of full rank and $B_{11}$ is of full rank, so
\[
\begin{bmatrix}
P B_{11} \\
P B_{21}
\end{bmatrix}
\text{ and } \begin{bmatrix}
P \\
P B_{21} B_{11}^{-1}
\end{bmatrix}
\text{ are of full column rank and } [Q \ B_{11}^{-1} B_{12} Q]
\text{ and } [B_{11} Q \ B_{12} Q] \text{ are of full row rank. So (**) and (***) give two factorizations of } A \text{ in the form } A = LR \text{ where } L \text{ is a } n \times r \text{ full column rank matrix and } R \text{ is a } r \times m \text{ full row rank.}
\]

**Definition.** If $n \times m$ matrix $A$ of rank $r$ can be written as $A = LR$ where $L$ is a $n \times r$ full column rank matrix and $R$ is a $r \times m$ full row rank matrix, then $A = LR$ is a full rank factorization of $A$.

**Theorem 3.4.1.** If $A$ is a square nonsingular matrix and $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where both $A_{11}$ and $A_{22}$ are nonsingular then in terms of the Schur complement of $A_{11}$ in $A$, $Z = A_{22} - A_{21} A_{11}^{-1} A_{12}$, we have that the inverse of $A$ is
\[
A^{-1} = \begin{bmatrix}
A_{11}^{-1} + A_{11}^{-1} A_{12} Z^{-1} A_{21} A_{11}^{-1} & -A_{11}^{-1} A_{12} Z^{-1} \\
-Z^{-1} A_{21} A_{11}^{-1} & Z^{-1}
\end{bmatrix}.
\]

**Note.** The proof of Theorem 3.4.1 is to be given in Exercise 3.13.

**Theorem 3.4.2.** If $A$ is a square matrix such that $A = \begin{bmatrix} X^T \\ y^T \end{bmatrix} [X \ y]$ where $X$ is of full column rank, then the Schur complement of $X^T X$ in $A$ is
\[
y^T y - y^T X (X^T X)^{-1} X^T y.
\]
Theorem 3.4.3. If $A$ is a square matrix partitioned as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where $A_{11}$ is square and nonsingular then

$$\det(A) = \det(A_{11}) \det(A_{22} - A_{21}A_{11}^{-1}A_{12}) = \det(A_{11}) \det(Z)$$

where $Z = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is the Schur complement of $A_{11}$ in $A$.  

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