

Dr. Bob's Axiom of Choice Centennial Lecture

Fall 2008 (updated Fall 2020)



Ernst Zermelo, 1871–1953 (from [Ernst Zermelo and Freiburg](#))

A Century Ago

Note. This year (2008) marks the 100 year anniversary of Ernst Zermelo's first statement of the currently accepted axioms of set theory. In particular, in 1908 he published two foundational works:

1. Neuer Beweis für die Möglichkeit einer Wohlordnung (A New Proof of the Possibility of a Well-Ordering), *Mathematische Annalen* **65**, 107–128 (translated into English in *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, Jean van Heijenoort, Cambridge: Harvard University Press, 1967, 183–198).
2. Untersuchungen über die Grundlagen der Mengenlehre, I (Investigations in the Foundations of Set Theory I), *Mathematische Annalen* **65**, 261–281 (translated into English in *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, Jean van Heijenoort, Cambridge: Harvard University Press, 1967, 199–215).

Note. In the first of these two papers, Zermelo addresses criticism of his 1904 paper “Beweis, dass jede Menge wohlgeordnet werden kann” (Proof that Every Set can be Well-Ordered), *Mathematische Annalen* **59**, 514–516 (translated into English in *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, Jean van Heijenoort, Cambridge: Harvard University Press, 1967, 139–141). In the 1904 paper, Zermelo uses the “Axiom of Choice” to prove that every set can be well-ordered (to be explained shortly). However, the mathematical community reacted negatively to Zermelo’s paper and he was motivated to respond in his first 1908 paper.

Zermelo’s Axioms of Set Theory

Note. In the second of these two papers, Zermelo presents the first axiomatic set theory. In that era, set theory was undergoing a bit of turmoil. Russell’s Paradox had been revealed and the foundations of set theory were coming into question. Part of Zermelo’s paper involves constructions of sets and his approach does not allow the constructions of sets which are “too big” (Russell, at roughly the same time, introduces his “theory of types” to address this problem). In addition, Zermelo seems to be the first to see that the existence of infinite sets must be taken as an axiom (page 199 of van Heijenoort).

Note. Zermelo states 7 axioms (to paraphrase):

1. Axiom of Extensionality.

If $M \subset N$ and $N \subset M$, then $M = N$.

2. Axiom of Elementary Sets.

There exists the null set \emptyset that contains no elements. For any object a in the universal set, the set $\{a\}$ exists. If a and b are in the universal set, the set $\{a, b\}$ exists.

3. Axiom of Separation.

For a proposition $P(x)$ with a truth value for all $x \in M$, M has a subset M_P containing as elements those elements x of M for which $P(x)$ is true.

4. Axiom of the Power Set.

For every set T there corresponds another set $\mathcal{P}(T)$, the *power set* of T , that contains as elements precisely all subsets of T .

5. Axiom of Union.

For every set T there corresponds another set $\cup T$, the *union* of T , that contains as elements precisely all elements of the elements of T . (If $T = \{A, B, C\}$ where A, B, C are sets, then $\cup T = A \cup B \cup C$.)

6. Axiom of Choice.

“If T is a set whose elements all are sets that are different from \emptyset and mutually disjoint, its union $\cup T$ includes at least one subset S_1 having one and only one element in common with each element of T .”

7. Axiom of Infinity.

“There exists at least one set Z that contains the null set as an element and that for any $a \in Z$, $\{a\} \in Z$.” (So Z is an infinite set since $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \dots\} \subset Z$. Z is called an *inductive* set.)

The Axiom of Choice

Note. We concentrate on the Axiom of Choice. Another statement of it is [Jech, 1973]:

Axiom of Choice. “For every family \mathcal{F} of nonempty sets, there is a function f such that $f(S) \in S$ for each set S in family \mathcal{F} . f is called a *choice function*.”

Some object to the Axiom of Choice due to its nonconstructive nature—it guarantees that an element can be chosen from every set in a family of sets, but it gives no constructive way to actually *choose* the elements. For example, if $\mathcal{F} = \{X, Y\}$ where X and Y are nonempty disjoint sets of real numbers, how would one choose an element of X and an element of Y ?

Note. The Axiom of Choice can be used to construct a (in the sense of Lebesgue) nonmeasurable set. A related result is the Banach-Tarski paradox which uses the Axiom of Choice to decompose a sphere into five pieces, two of which can be combined (through rigid rotations) to give a copy of the sphere and the other three which can be (rigidly) combined to give a second copy of the sphere. That is, the sphere can be cut into pieces which can then be combined to make two copies of the sphere, thus doubling the volume of the sphere out of nothing!

Note. We now consider 3 “principles” equivalent to the Axiom of Choice:

1. **Well-Ordering Principle (Zermelo’s Theorem).**

Every set can be well-ordered.

2. **Maximal Principle I (Zorn’s Lemma).**

Let $(P, <)$ be a nonempty partially ordered set and let every chain in P have an upper bound. Then P has a maximal element.

3. **Maximal Principle II (Tukey’s Lemma).**

Let \mathcal{F} be a nonempty family of sets. If \mathcal{F} has finite character, then \mathcal{F} has a maximal element (maximal with respect to set inclusion).

We now elaborate on these ideas.

Well-Orderings

Definition. [Hrbacek and Jech, 1984] A set R is a *binary relation* on set A if all elements of R are ordered pairs of elements of A . We denote $(a, b) \in R$ as aRb . A binary relation R in A which is reflexive (for all $a \in A$, aRa), antisymmetric (for all $a, b \in A$ with aRb and bRa , we have $a = b$), and transitive (for all $a, b, c \in A$ with aRb and bRc , we have aRc) is said to be a *partial ordering* of A . We denote such an ordering as \leq . If for $a, b \in A$, either $a \leq b$ or $b \leq a$ then a and b are *comparable*. An ordering \leq of A is a *total ordering* (or *linear ordering*) if any two elements of A are comparable.

Example. Subset inclusion \subset is a partial ordering on, say, $\mathcal{P}(\mathbb{R})$. Certain sets are not comparable under this partial ordering (for example, $(1, 3)$ and $(0, 2)$), so this is not a total ordering.

Example. Less than or equal to \leq is a total ordering on \mathbb{R} .

Definition. A total ordering \prec of a set A is a *well-ordering* if every nonempty subset of A has a \prec -least element.

Example. Less than or equal to \leq is a well-ordering on \mathbb{N} . However, \leq is not a well-ordering of \mathbb{R} (consider the nonempty subset $A = (0, 1)$ of \mathbb{R}).

Note. The Well-Ordering Principle says that there is a well-ordering of every set.

Note. You may have heard that there is no ordering of the complex numbers. What this means is that there is no way to define “ \leq ” on \mathbb{C} in such a way that the usual ordering on \mathbb{R} comes as a “subordering.” This idea of well-ordering a set should not be confused with ordering of a field (in, for example, the definition of \mathbb{R} as a “complete ordered field”).

Chains

Definition. A subset C of a partially ordered set (P, \prec) is a *chain* in P if C is totally ordered by \prec . $u \in P$ is an *upper bound* of chain C if $c \prec u$ for all $c \in C$. $a \in P$ is a *maximal element* if $a \prec x$ for no $x \in P$.

Note. The Maximal Principle I (Zorn's Lemma) states that a nonempty partially ordered set P for which every chain in P has an upper bound, has a maximal element.

Finite Character

Definition. Let \mathcal{F} be a family of sets. We say that \mathcal{F} has *finite character* if for each set X , $X \in \mathcal{F}$ if and only if every finite subset of X belongs to \mathcal{F} .

Note. The Maximal Principle II (Tukey's Lemma) states that every nonempty family of sets of finite character has a maximal element (maximal with respect to subset inclusion).

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Revised: 9/26/2020