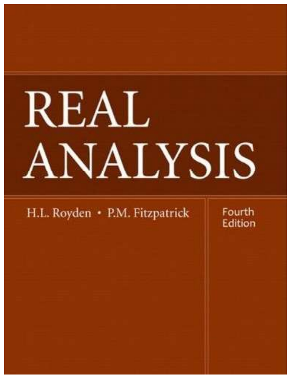
Real Analysis

Chapter 1. The Real Numbers: Sets, Sequences, and Functions

1.4. Open Sets, Closed Sets, and Borel Sets or Real Numbers—Proofs of Theorems



Theorem 1.4.A

Theorem 1.4.A. Given any collection $\mathcal{C}$ of subsets of X, there exists a smallest algebra $\mathcal{A}$ which contains $\mathcal{C}$. That is, if $\mathcal{B}$ is any algebra containing $\mathcal{C}$ then $\mathcal{B}$ contains $\mathcal{A}$.

Proof. Let $\mathcal{F}$ be the family of all algebras $\mathcal{B}$ of X that contain $\mathcal{C}$ (the power set $\mathcal{P}(X) = 2^X \in \mathcal{F}$, so $\mathcal{F} \neq \varnothing$). Let $\mathcal{A} = \cap_{\mathcal{B} \in \mathcal{F}} \mathcal{B}$.

Since $\mathcal{C} \subset \mathcal{B}$ for all $\mathcal{B} \in \mathcal{F}$, then $\mathcal{C} \subset \mathcal{A}$ and $\mathcal{A}$ contains $\mathcal{C}$.

If $A, B \in \mathcal{A}$, then $A, B \in \mathcal{B}$ for all $\mathcal{B} \in \mathcal{F}$ and so $A \cup B \in \mathcal{A}$ since each $\mathcal{B}$ is an algebra. Therefore $\mathcal{A}$ is closed under finite unions.

Similarly, if $A \in \mathcal{A}$, then $\tilde{A} = X \sim A = X \setminus A \in \mathcal{B}$ for all $\mathcal{B} \in \mathcal{F}$ and so $\tilde{A} \in \mathcal{A}$. Therefore $\mathcal{A}$ is closed under complements and so $\mathcal{A}$ is an algebra.

Now, if $\mathcal{B}$ is an algebra containing $\mathcal{C}$ then $\mathcal{B} \in \mathcal{F}$ and, by definition, $\mathcal{B} \supset \mathcal{A}$. So $\mathcal{A}$ is "the smallest" algebra containing collection $\mathcal{C}$. □

Proposition 1.13

Proposition 1.13. Let $\mathcal{C}$ be a collection of subsets of a set X. Then the intersection $\mathcal{A}$ of all σ-algebras of subsets of X that contain $\mathcal{C}$ is a σ-algebra that contains $\mathcal{C}$. Moreover, it is the smallest σ-algebra of subsets of X that contain $\mathcal{C}$ in the sense that if $\mathcal{B}$ is a σ-algebra containing $\mathcal{C}$, then $\mathcal{A} \subset \mathcal{B}$.

Proof. We know by Theorem 1.4.A above that the intersection of algebras containing $\mathcal{C}$ are again algebras. So we only need to prove the "σ" part and the "smallest" part. We leave this as homework. □