Chapter 1. The Real Numbers: Sets, Sequences, and Functions
1.4. Open Sets, Closed Sets, and Borel Sets or Real Numbers—Proofs of Theorems
Table of contents

1. Theorem

2. Proposition 1.13
Theorem. Given any collection $C$ of subsets of $X$, there exists a smallest algebra $A$ which contains $C$. That is, if $B$ is any algebra containing $C$ then $B$ contains $A$.

Proof. Let $\mathcal{F}$ be the family of all algebras $B$ of $X$ that contain $C$ (the power set $\mathcal{P}(X) = 2^X \in \mathcal{F}$, so $\mathcal{F} \neq \emptyset$). Let $A = \cap_{B \in \mathcal{F}} B$. 
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If \( A, B \in A \), then \( A, B \in B \) for all \( B \in F \) and so \( A \cup B \in A \) since each \( B \) is an algebra. Therefore \( A \) is closed under finite unions.
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Similarly, if $A \in A$, then $\tilde{A} = X \sim A = X \setminus A \in B$ for all $B \in \mathcal{F}$ and so $\tilde{A} \in A$. Therefore $A$ is closed under complements and so $A$ is an algebra.
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Since \( C \subset B \) for all \( B \in \mathcal{F} \), then \( C \subset A \) and \( A \) contains \( C \).

If \( A, B \in A \), then \( A, B \in B \) for all \( B \in \mathcal{F} \) and so \( A \cup B \in A \) since each \( B \) is an algebra. Therefore \( A \) is closed under finite unions.

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Now, if \( B \) is an algebra containing \( C \) then \( B \in \mathcal{F} \) and, by definition, \( B \supset A \). So \( A \) is “the smallest” algebra containing collection \( C \).
Proposition 1.13. Let $C$ be a collection of subsets of a set $X$. Then the intersection $A$ of all $\sigma$-algebras of subsets of $X$ that contain $C$ is a $\sigma$-algebra that contains $C$. Moreover, it is the smallest $\sigma$-algebra of subsets of $X$ that contain $C$ in the sense that if $B$ is a $\sigma$-algebra containing $C$, then $A \subset B$.

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