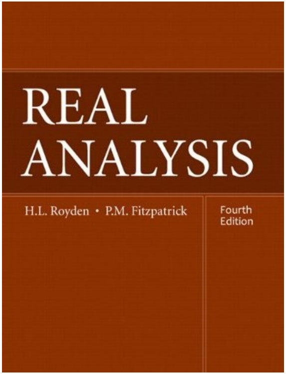
Real Analysis

Chapter 1. The Real Numbers: Sets, Sequences, and Functions

1.4. Open Sets, Closed Sets, and Borel Sets or Real Numbers—Proofs of Theorems



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