Real Analysis

Chapter 17. General Measure Spaces: Their Properties and Construction

17.1. Measures and Measurable Sets—Proofs of Theorems

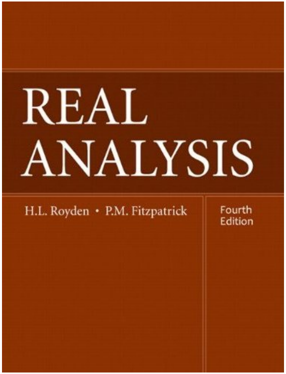

Table of contents

1. Proposition 17.1
2. The Borel-Cantelli Lemma

Proposition 17.1

Proposition 17.1. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) For any finite disjoint collection $\{E_k\}_{k=1}^n$ of measurable sets,
$$\mu\left(\dot{\bigcup_{k=1}^n} E_k\right) = \sum_{k=1}^n \mu(E_k).$$
That is, μ is *finite additive*.

(ii) If A and B are measurable sets and $A \subseteq B$, then $\mu(A) \leq \mu(B)$. That is, μ is *monotone*.

(iii) If A and B are measurable sets, $A \subseteq B$, and $\mu(A) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$. This is the *excision principle*.

(iv) For any countable collection $\{E_k\}_{k=1}^\infty$ of measurable sets that covers a measurable set E,
$$\mu(E) \leq \sum_{k=1}^\infty \mu(E_k).$$
This is called *countable monotonicity*.

Proposition 17.1 (continued)

Proof. (i) Finite Additivity follows from countable additivity by taking $E_k = \varnothing$ for $k > n$.

(ii, iii) By finite additivity we have $\mu(B) = \mu(A) + \mu(B \setminus A)$ and since $\mu(B \setminus A) \geq 0$, monotonicity follows. Rearranging this equation gives the excision principle.

Proposition 17.1 (continued)

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(iv) Define $G_1 = E_1$ and $G_k = E_k \setminus \left(\cup_{i=1}^{k-1} E_i\right)$ for $k \geq 2$. Then $\{G_k\}_{k=1}^{\infty}$ is a sequence of disjoint sets, and $\cup_{k=1}^{\infty} G_k = \cup_{k=1}^{\infty} E_k$. Also, $G_k \subset E_k$ for all $k \in \mathbb{N}$. So

$$\begin{aligned} \mu(E) &\leq \mu\left(\cup_{k=1}^{\infty} E_k\right) \text{ by monotonicity} \\ &= \mu\left(\dot\cup_{k=1}^{\infty} G_k\right) = \sum_{k=1}^{\infty} \mu(G_k) \text{ by countable additivity} \\ &\leq \sum_{k=1}^{\infty} \mu(E_k) \text{ by monotonicity.} \end{aligned}$$

□

Proposition 17.1 (continued)

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The Borel-Cantelli Lemma

The Borel-Cantelli Lemma. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\displaystyle\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Then almost all $x \in X$ belong to at most a finite number of the E_k's.

Proof. For $n \in \mathbb{N}$, countable monotonicity implies $\mu\left(\cup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu(E_k)$.

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Proof. For $n \in \mathbb{N}$, countable monotonicity implies $\mu\left(\cup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu(E_k)$. Hence, Continuity of Measure (Proposition 17.2), since $\cup_{k=n}^{\infty} E_k$ is a descending sequence of sets, gives

$$\begin{aligned} \mu\left(\cap_{n=1}^{\infty}\left[\cup_{k=n}^{\infty} E_k\right]\right) &= \lim_{n\to\infty} \mu\left(\cup_{k=n}^{\infty} E_k\right) \\ &\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(E_k) \text{ by above} \\ &= 0 \text{ since the tail of a convergent series} \\ &\quad \text{of real numbers goes to } 0 \end{aligned}$$

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The Borel-Cantelli Lemma (continued)

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Proof (continued). Explicitly $\cap_{n=1}^{\infty} (\cup_{k=n}^{\infty} E_k)$ is the set of all points in X which belong to an infinite number of E_k's. □