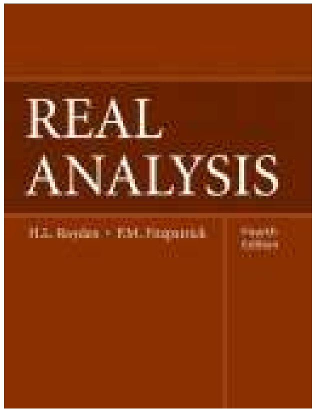
Real Analysis

Chapter 17. General Measure Spaces: Their Properties and Construction

17.2. Signed Measures: Hahn and Jordan Decompositions—Proofs



Proposition 17.4

Proposition 17.4. Let ν be a signed measure on $(X, \mathcal{M})$. Then the union of a countable collection of positive sets is positive.

Proof. Let $A = \cup_{k=1}^{\infty} A_K$ and $E \subset A$, where $A, E, A_k \in \mathcal{M}$ for all $k \in \mathbb{N}$. Define $E_1 = E \cap A_1$ and for $k \geq 2$ define

$$E_k = (E \cap A_k) \setminus (A_1 \cup A_2 \cup \cdots \cup A_{k-1}).$$

Then for each $E_k \in \mathcal{M}$ and $\nu(E_k) \geq 0$ since A_k is positive. $\{E_k\}_{k=1}^{\infty}$ is a disjoint collection, $E = \dot\cup_{k=1}^{\infty} E_k$, and so $\nu(E) = \sum_{k=1}^{\infty} \nu(E_k) \geq 0$. So A is positive. □

Hahn's Lemma

Hahn's Lemma. Let ν be a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$ where $0 < \nu(E) < \infty$. Then there is $A \subset E$, $A \in \mathcal{M}$ that is positive and of positive measure.

Proof. If E is positive, then we are done. If E is not positive, then E contains subsets of negative measure. Let m_1 be the smallest natural number for which there is a measurable set of measure less than $-1/m_1$. Let $E_1 \subset E$ with $\nu(E_1) < -1/m_1$. Inductively define natural numbers $m_1, m_2, \ldots, m_n$ and measurable sets $E_1, E_2, \ldots, E_n$ such that, for $1 \leq k \leq n$, m_k is the smallest natural number for which there is a measurable subset of $E \setminus \dot\cup_{j=1}^{k-1} E_j$ of measure less than $-1/m_k$ and E_k is a subset of $E \setminus \dot\cup_{j=1}^{k-1} E_j$ for which $\nu(E_k) < -1/m_k$. If the process terminates at some $n \in \mathbb{N}$, then set $A = E \setminus \dot\cup_{j=1}^{n} E_j$ is a positive subset of E. If the process does not terminate, define $A = E \setminus \dot\cup_{k=1}^{\infty} E_k$. Then $E = A \dot\cup (\dot\cup_{k=1}^{\infty} E_k)$.

Hahn's Lemma (continued 1)

Hahn's Lemma. Let ν be a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$ where $0 < \nu(E) < \infty$. Then there is $A \subset E$, $A \in \mathcal{M}$ that is positive and of positive measure.

Proof (continued). Since $\dot\cup_{k=1}^{\infty} E_k \in \mathcal{M}$ and $\dot\cup_{k=1}^{\infty} E_k \subset E$, then (by Lemma 1 and countable additivity)

$$-\infty < \nu\left(\dot\cup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k) \leq \sum_{k=1}^{\infty} -1/m_k.$$

So $m_k \to \infty$ (otherwise, if m_k converges, then the series on the right would diverge). Now to show that A is positive. Let $B \subset A$ be measurable. Then $B \subset A \subset E \setminus \left(\dot\cup_{j=1}^{k-1} E_j\right)$ for each $k \in \mathbb{N}$. Since m_k is the smallest natural number such that there is a measurable subset of $E \setminus \left(\dot\cup_{j=1}^{k-1} E_j\right)$ of measure less than $-1/m_k$ (so $-1/(m_k - 1) < \nu(E_k) \leq -1/m_k$), then it must be that $\nu(B) > -1/(m_k - 1)$. Since this holds for all $k \in \mathbb{N}$ and $m_k \to \infty$, then $\nu(B) \geq 0$. So A is a positive set.

Hahn's Lemma (continued 2)

Hahn's Lemma. Let ν be a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$ where $0 < \nu(E) < \infty$. Then there is $A \subset E$, $A \in \mathcal{M}$ that is positive and of positive measure.

Proof (continued). Finally, $E = A \,\dot\cup\, (\cup_{k=1}^{\infty} E_k)$ (or $E = A \,\dot\cup\, (\cup_{k=1}^{n} E_k)$ in the first case), so $\nu(E) = \nu(A) + \nu(\cup_{k=1}^{\infty} E_k) > 0$ and since $\nu(\cup_{k=1}^{\infty} E_k) < 0$, it must be that $\nu(A) > 0$. □

Hahn Decomposition Theorem

The Hahn Decomposition Theorem. Let ν be a signed measure on $(X, \mathcal{M})$. Then there is a Hahn decomposition of X.

Proof. Without loss of generality, suppose $+\infty$ is the infinite value omitted by ν (otherwise, replace ν with $-\nu$ and follow this proof). Let $\mathcal{P}$ be the collection of positive subsets of X and define $\lambda = \sup\{\nu(E) \mid E \in \mathcal{P}\}$. Then $\lambda \geq 0$ since $\varnothing \in \mathcal{P}$. Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of positive sets such that $\lambda = \lim_{k\to\infty} \nu(A_k)$ (which exists by the definition of supremum). Define $A = \cup_{k=1}^{\infty} A_k$. By Proposition 17.4, set A is a positive set, and so $\lambda \geq \nu(A)$ (by the definition of supremum). Also, for each $k \in \mathbb{N}$, $A \setminus A_n \subset A$ and so $\nu(A \setminus A_k) \geq 0$ since A is positive. Thus $\nu(A) = \nu(A_k) + \nu(A \setminus A_k) \geq \nu(A_k)$. Hence $\nu(A) \geq \lambda$. Therefore $\nu(A) = \lambda$, and $\lambda < \infty$ since λ does not take on the value $+\infty$ (WLOG).

Hahn Decomposition Theorem (continued)

The Hahn Decomposition Theorem. Let ν be a signed measure on $(X, \mathcal{M})$. Then there is a Hahn decomposition of X.

Proof (continued). Let $B = X \setminus A$. ASSUME B is not negative. Then there is a subset E of B with positive measure. So by Hahn's Lemma there is $E_0 \subset B$ such that E_0 is positive and $\nu(E_0) > 0$. But then $A \,\dot\cup\, E_0$ is a positive set by Proposition 17.4 and by additivity, $\nu(A \,\dot\cup\, E_0) = \nu(A) + \nu(E_0) > \lambda$, a CONTRADICTION to the definition of λ (notice that $\lambda < \infty$ is needed here). So the assumption that B is not negative is false and hence B is a negative set. Therefore $\{A, B\}$ is a Hahn decomposition of X. □

Jordan Decomposition Theorem

The Jordan Decomposition Theorem.
Let ν be a signed measure on $(X, \mathcal{M})$. Then there are two mutually singular measures ν^+ and ν^- on $(X, \mathcal{M})$ for which $\nu = \nu^+ - \nu^-$. Moreover, there is only one such pair of mutually singular measures.

Proof. Let $\{A, B\}$ be a Hahn decomposition of X, which exists by the Hahn Decomposition Theorem. Then for $E \in \mathcal{M}$, define $\nu^+(E) = \nu(E \cap A)$ and $\nu^-(E) = -\nu(E \cap B)$. Then $\nu^+(B) = 0$ and $\nu^-(A) = 0$, so ν^+ and ν^- are mutually singular. Also,

$$\begin{aligned} \nu(E) &= \nu((E \cap A) \,\dot\cup\, (E \cap B)) \\ &= \nu(E \cap A) + \nu(E \cap B) \text{ by additivity} \\ &= \nu^+(E) - \nu^-(E), \end{aligned}$$

so $\nu = \nu^+ - \nu^-$.

Jordan Decomposition Theorem (continued)

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Proof. For uniqueness is given in Problem 17.13. □