Chapter 17. General Measure Spaces: Their Properties and Construction

17.2. Signed Measures: Hahn and Jordan Decompositions—Proofs
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Proposition 17.4

**Proposition 17.4.** Let $\nu$ be a signed measure on $(X, \mathcal{M})$. Then the union of a countable collection of positive sets is positive.

**Proof.** Let $A = \bigcup_{k=1}^{\infty} A_k$ and $E \subset A$, where $A, E, A_k \in \mathcal{M}$ for all $k \in \mathbb{N}$. Define $E_1 = E \cap A_1$ and for $k \geq 2$ define

$$E_k = (E \cap A_k) \setminus (A_1 \cup A_2 \cup \cdots \cup A_{k-1}).$$
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Then for each $E_k \in \mathcal{M}$ and $\nu(E_k) \geq 0$ since $A_k$ is positive. \{E_k\}_{k=1}^{\infty} is a disjoint collection, $E = \bigcup_{k=1}^{\infty} E_k$, and so $\nu(E) = \sum_{k=1}^{\infty} \nu(E_k) \geq 0$. So $A$ is positive. \qed
Proposition 17.4. Let $\nu$ be a signed measure on $(X, \mathcal{M})$. Then the union of a countable collection of positive sets is positive.

Proof. Let $A = \bigcup_{k=1}^{\infty} A_k$ and $E \subset A$, where $A, E, A_k \in \mathcal{M}$ for all $k \in \mathbb{N}$. Define $E_1 = E \cap A_1$ and for $k \geq 2$ define

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Hahn’s Lemma. Let $\nu$ be a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$ where $0 < \nu(E) < \infty$. Then there is $A \subset E$, $A \in \mathcal{M}$ that is positive and of positive measure.

Proof. If $E$ is positive, then we are done. If $E$ is not positive, then $E$ contains subsets of negative measure. Let $m_1$ be the smallest natural number for which there is a measurable set of measure less than $-1/m_1$. Let $E_1 \subset E$ with $\nu(E_1) < -1/m_1$. 
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Proof. If $E$ is positive, then we are done. If $E$ is not positive, then $E$ contains subsets of negative measure. Let $m_1$ be the smallest natural number for which there is a measurable set of measure less than $-1/m_1$. Let $E_1 \subset E$ with $\nu(E_1) < -1/m_1$. Inductively define natural numbers $m_1, m_2, \ldots, m_n$ and measurable sets $E_1, E_2, \ldots, E_n$ such that, for $1 \leq k \leq n$, $m_k$ is the smallest natural number for which there is a measurable subset of $E \setminus \bigcup_{j=1}^{k-1} E_j$ of measure less than $-1/m_k$ and $E_k$ is a subset of $E \setminus \bigcup_{j=1}^{k-1} E_j$ for which $\nu(E_k) < -1/m_k$. If the process terminates at some $n \in \mathbb{N}$, then set $A = E \setminus \bigcup_{j=1}^{n} E_j$ is a positive subset of $E$. 
Hahn’s Lemma. Let $\nu$ be a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$ where $0 < \nu(E) < \infty$. Then there is $A \subset E$, $A \in \mathcal{M}$ that is positive and of positive measure.

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Hahn’s Lemma (continued 1)

**Hahn’s Lemma.** Let $\nu$ be a signed measure on $(X, M)$ and $E \in M$ where $0 < \nu(E) < \infty$. Then there is $A \subset E$, $A \in M$ that is positive and of positive measure.

**Proof (continued).** Since $\bigcup_{k=1}^{\infty} E_k \in M$ and $\bigcup_{k=1}^{\infty} E_k \subset E$, then (by Lemma 1 and countable additivity)

$$-\infty < \nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k) \leq \sum_{k=1}^{\infty} -1/m_k.$$

So $m_k \to \infty$ (otherwise, if $m_k$ converges, then the series on the right would diverge). Now to show that $A$ is positive. Let $B \subset A$ be measurable. Then $B \subset A \subset E \setminus \left(\bigcup_{j=1}^{k-1} E_j\right)$ for each $k \in \mathbb{N}$. Since $m_k$ is the smallest natural number such that there is a measurable subset of $E \setminus \left(\bigcup_{j=1}^{k-1} E_j\right)$ of measure less than $-1/m_k$ (so $-1/(m_k - 1) < \nu(E_k) \leq -1/m_k$), then it must be that $\nu(B) > -1/(m_k - 1)$. 
Hahn’s Lemma. Let $\nu$ be a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$ where $0 < \nu(E) < \infty$. Then there is $A \subset E$, $A \in \mathcal{M}$ that is positive and of positive measure.

Proof (continued). Since $\bigcup_{k=1}^{\infty} E_k \in \mathcal{M}$ and $\bigcup_{k=1}^{\infty} E_k \subset E$, then (by Lemma 1 and countable additivity)

$$-\infty < \nu \left( \bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} \nu(E_k) \leq \sum_{k=1}^{\infty} -1/m_k.$$ 

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Hahn’s Lemma (continued 1)

**Hahn’s Lemma.** Let \( \nu \) be a signed measure on \((X, \mathcal{M})\) and \( E \in \mathcal{M} \) where \( 0 < \nu(E) < \infty \). Then there is \( A \subset E, A \in \mathcal{M} \) that is positive and of positive measure.

**Proof (continued).** Since \( \bigcup_{k=1}^{\infty} E_k \in \mathcal{M} \) and \( \bigcup_{k=1}^{\infty} E_k \subset E \), then (by Lemma 1 and countable additivity)

\[
-\infty < \nu \left( \bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} \nu(E_k) \leq \sum_{k=1}^{\infty} -1/m_k.
\]

So \( m_k \to \infty \) (otherwise, if \( m_k \) converges, then the series on the right would diverge). Now to show that \( A \) is positive. Let \( B \subset A \) be measurable. Then \( B \subset A \subset E \setminus \left( \bigcup_{j=1}^{k-1} E_j \right) \) for each \( k \in \mathbb{N} \). Since \( m_k \) is the smallest natural number such that there is a measurable subset of \( E \setminus \left( \bigcup_{j=1}^{k-1} E_j \right) \) of measure less than \(-1/m_k\) (so \(-1/(m_k - 1) < \nu(E_k) \leq -1/m_k\)), then it must be that \( \nu(B) > -1/(m_k - 1) \). Since this holds for all \( k \in \mathbb{N} \) and \( m_k \to \infty \), then \( \nu(B) \geq 0 \). So \( A \) is a positive set.
Hahn’s Lemma. Let \( \nu \) be a signed measure on \((X, \mathcal{M})\) and \( E \in \mathcal{M} \) where \( 0 < \nu(E) < \infty \). Then there is \( A \subset E, A \in \mathcal{M} \) that is positive and of positive measure.

Proof (continued). Finally, \( E = A \cup (\bigcup_{k=1}^{\infty} E_k) \) (or \( E = A \cup (\bigcup_{k=1}^{n} E_k) \) in the first case), so \( \nu(E) = \nu(A) + \nu(\bigcup_{k=1}^{\infty} E_k) > 0 \) and since \( \nu(\bigcup_{k=1}^{\infty} E_k) < 0 \), it must be that \( \nu(A) > 0 \). \( \square \)
The Hahn Decomposition Theorem. Let $\nu$ be a signed measure on $(X, \mathcal{M})$. Then there is a Hahn decomposition of $X$.

Proof. Without loss of generality, suppose $+\infty$ is the infinite value omitted by $\nu$ (otherwise, replace $\nu$ with $-\nu$ and follow this proof).
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\[
\lambda = \sup\{\nu(E) \mid E \in \mathcal{P}\}.
\]
Then \( \lambda \geq 0 \) since \( \emptyset \in \mathcal{P} \). Let \( \{A_k\}_{k=1}^{\infty} \) be a sequence of positive sets such that \( \lambda = \lim_{k \to \infty} \nu(A_k) \) (which exists by the definition of supremum). Define \( A = \bigcup_{k=1}^{\infty} A_k \).
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\[
\nu(A) = \nu(A_k) + \nu(A \setminus A_k) \geq \nu(A_k).
\]

Hence \( \nu(A) \geq \lambda \). Therefore \( \nu(A) = \lambda \), and \( \lambda < \infty \) since \( \lambda \) does not take on the value \(+\infty\) (WLOG).
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Proof (continued). Let $B = X \setminus A$. ASSUME $B$ is not negative. Then there is a subset $E$ of $B$ with positive measure. So by Hahn’s Lemma there is $E_0 \subset B$ such that $E_0$ is positive and $\nu(E_0) > 0$. 
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The Jordan Decomposition Theorem.
Let $\nu$ be a signed measure on $(X, \mathcal{M})$. Then there are two mutually singular measures $\nu^+$ and $\nu^-$ on $(X, \mathcal{M})$ for which $\nu = \nu^+ - \nu^-$. Moreover, there is only one such pair of mutually singular measures.

Proof. Let $\{A, B\}$ be a Hahn decomposition of $X$, which exists by the Hahn Decomposition Theorem. Then for $E \in \mathcal{M}$, define $\nu^+(E) = \nu(E \cap A)$ and $\nu^-(E) = -\nu(E \cap B)$. Then $\nu^+(B) = 0$ and $\nu^-(A) = 0$, so $\nu^+$ and $\nu^-$ are mutually singular. Also,

\[
\nu(E) = \nu((E \cap A) \cup (E \cap B)) \\
= \nu(E \cap A) + \nu(E \cap B) \text{ by additivity} \\
= \nu^+(E) - \nu^-(E),
\]

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Proof. For uniqueness is given in Problem 17.13.