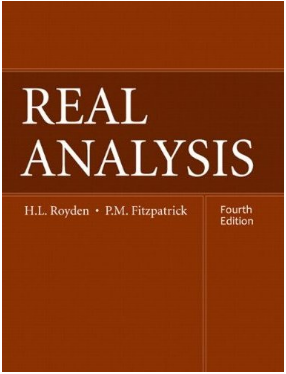
# Real Analysis

## Chapter 17. General Measure Spaces: Their Properties and Construction

17.2. Signed Measures: Hahn and Jordan Decompositions—Proofs



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$$E_k = (E \cap A_k) \setminus (A_1 \cup A_2 \cup \cdots \cup A_{k-1}).$$

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Then for each $E_k \in \mathcal{M}$ and $\nu(E_k) \geq 0$ since $A_k$ is positive. $\{E_k\}_{k=1}^{\infty}$ is a disjoint collection, $E = \dot{\cup}_{k=1}^{\infty} E_k$, and so $\nu(E) = \sum_{k=1}^{\infty} \nu(E_k) \geq 0$. So $A$ is positive. □

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# Hahn's Lemma

**Hahn's Lemma.** Let $\nu$ be a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$ where $0 < \nu(E) < \infty$. Then there is $A \subset E$, $A \in \mathcal{M}$ that is positive and of positive measure.

**Proof.** If $E$ is positive, then we are done. If $E$ is not positive, then $E$ contains subsets of negative measure. Let $m_1$ be the smallest natural number for which there is a measurable set of measure less than $-1/m_1$. Let $E_1 \subset E$ with $\nu(E_1) < -1/m_1$.

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**Proof.** If $E$ is positive, then we are done. If $E$ is not positive, then $E$ contains subsets of negative measure. Let $m_1$ be the smallest natural number for which there is a measurable set of measure less than $-1/m_1$. Let $E_1 \subset E$ with $\nu(E_1) < -1/m_1$. Inductively define natural numbers $m_1, m_2, \ldots, m_n$ and measurable sets $E_1, E_2, \ldots, E_n$ such that, for $1 \leq k \leq n$, $m_k$ is the smallest natural number for which there is a measurable subset of $E \setminus \cup_{j=1}^{k-1} E_j$ of measure less than $-1/m_k$ and $E_k$ is a subset of $E \setminus \cup_{j=1}^{k-1} E_j$ for which $\nu(E_k) < -1/m_k$. If the process terminates at some $n \in \mathbb{N}$, then set $A = E \setminus \cup_{j=1}^{n} E_j$ is a positive subset of $E$.

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# Hahn's Lemma (continued 1)

**Hahn's Lemma.** Let $\nu$ be a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$ where $0 < \nu(E) < \infty$. Then there is $A \subset E$, $A \in \mathcal{M}$ that is positive and of positive measure.

**Proof (continued).** Since $\dot\cup_{k=1}^{\infty} E_k \in \mathcal{M}$ and $\dot\cup_{k=1}^{\infty} E_k \subset E$, then (by Lemma 1 and countable additivity)

$$-\infty < \nu\left(\dot\cup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k) \leq \sum_{k=1}^{\infty} -1/m_k.$$

So $m_k \to \infty$ (otherwise, if $m_k$ converges, then the series on the right would diverge). Now to show that $A$ is positive. Let $B \subset A$ be measurable. Then $B \subset A \subset E \setminus \left(\dot\cup_{j=1}^{k-1} E_j\right)$ for each $k \in \mathbb{N}$. Since $m_k$ is the smallest natural number such that there is a measurable subset of $E \setminus \left(\dot\cup_{j=1}^{k-1} E_j\right)$ of measure less than $-1/m_k$ (so $-1/(m_k - 1) < \nu(E_k) \leq -1/m_k$), then it must be that $\nu(B) > -1/(m_k - 1)$.

# Hahn's Lemma (continued 1)

**Hahn's Lemma.** Let $\nu$ be a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$ where $0 < \nu(E) < \infty$. Then there is $A \subset E$, $A \in \mathcal{M}$ that is positive and of positive measure.

**Proof (continued).** Since $\cup_{k=1}^{\infty} E_k \in \mathcal{M}$ and $\cup_{k=1}^{\infty} E_k \subset E$, then (by Lemma 1 and countable additivity)

$$-\infty < \nu\left(\cup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k) \leq \sum_{k=1}^{\infty} -1/m_k.$$

So $m_k \to \infty$ (otherwise, if $m_k$ converges, then the series on the right would diverge). Now to show that $A$ is positive. Let $B \subset A$ be measurable. Then $B \subset A \subset E \setminus \left(\cup_{j=1}^{k-1} E_j\right)$ for each $k \in \mathbb{N}$. Since $m_k$ is the smallest natural number such that there is a measurable subset of $E \setminus \left(\cup_{j=1}^{k-1} E_j\right)$ of measure less than $-1/m_k$ (so $-1/(m_k - 1) < \nu(E_k) \leq -1/m_k$), then it must be that $\nu(B) > -1/(m_k - 1)$. Since this holds for all $k \in \mathbb{N}$ and $m_k \to \infty$, then $\nu(B) \geq 0$. So $A$ is a positive set.

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**Proof (continued).** Since $\dot\cup_{k=1}^{\infty} E_k \in \mathcal{M}$ and $\dot\cup_{k=1}^{\infty} E_k \subset E$, then (by Lemma 1 and countable additivity)

$$-\infty < \nu\left(\dot\cup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k) \leq \sum_{k=1}^{\infty} -1/m_k.$$

So $m_k \to \infty$ (otherwise, if $m_k$ converges, then the series on the right would diverge). Now to show that $A$ is positive. Let $B \subset A$ be measurable. Then $B \subset A \subset E \setminus \left(\dot\cup_{j=1}^{k-1} E_j\right)$ for each $k \in \mathbb{N}$. Since $m_k$ is the smallest natural number such that there is a measurable subset of $E \setminus \left(\dot\cup_{j=1}^{k-1} E_j\right)$ of measure less than $-1/m_k$ (so $-1/(m_k - 1) < \nu(E_k) \leq -1/m_k$), then it must be that $\nu(B) > -1/(m_k - 1)$. Since this holds for all $k \in \mathbb{N}$ and $m_k \to \infty$, then $\nu(B) \geq 0$. So $A$ is a positive set.

# Hahn's Lemma (continued 2)

**Hahn's Lemma.** Let $\nu$ be a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$ where $0 < \nu(E) < \infty$. Then there is $A \subset E$, $A \in \mathcal{M}$ that is positive and of positive measure.

**Proof (continued).** Finally, $E = A \mathbin{\dot\cup} \left(\cup_{k=1}^{\infty} E_k\right)$ (or $E = A \mathbin{\dot\cup} \left(\cup_{k=1}^{n} E_k\right)$ in the first case), so $\nu(E) = \nu(A) + \nu\left(\cup_{k=1}^{\infty} E_k\right) > 0$ and since $\nu\left(\cup_{k=1}^{\infty} E_k\right) < 0$, it must be that $\nu(A) > 0$. □

# Hahn Decomposition Theorem

**The Hahn Decomposition Theorem.** Let $\nu$ be a signed measure on $(X, \mathcal{M})$. Then there is a Hahn decomposition of $X$.

**Proof.** Without loss of generality, suppose $+\infty$ is the infinite value omitted by $\nu$ (otherwise, replace $\nu$ with $-\nu$ and follow this proof).

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# Hahn Decomposition Theorem (continued)

**The Hahn Decomposition Theorem.** Let $\nu$ be a signed measure on $(X, \mathcal{M})$. Then there is a Hahn decomposition of $X$.

**Proof (continued).** Let $B = X \setminus A$. ASSUME $B$ is not negative. Then there is a subset $E$ of $B$ with positive measure. So by Hahn's Lemma there is $E_0 \subset B$ such that $E_0$ is positive and $\nu(E_0) > 0$.

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# Jordan Decomposition Theorem

**The Jordan Decomposition Theorem.**
Let $\nu$ be a signed measure on $(X, \mathcal{M})$. Then there are two mutually singular measures $\nu^+$ and $\nu^-$ on $(X, \mathcal{M})$ for which $\nu = \nu^+ - \nu^-$. Moreover, there is only one such pair of mutually singular measures.

**Proof.** Let $\{A, B\}$ be a Hahn decomposition of $X$, which exists by the Hahn Decomposition Theorem. Then for $E \in \mathcal{M}$, define $\nu^+(E) = \nu(E \cap A)$ and $\nu^-(E) = -\nu(E \cap B)$. Then $\nu^+(B) = 0$ and $\nu^-(A) = 0$, so $\nu^+$ and $\nu^-$ are mutually singular. Also,

$$\begin{aligned} \nu(E) &= \nu((E \cap A) \mathbin{\dot\cup} (E \cap B)) \\ &= \nu(E \cap A) + \nu(E \cap B) \text{ by additivity} \\ &= \nu^+(E) - \nu^-(E), \end{aligned}$$

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# Jordan Decomposition Theorem (continued)

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**Proof.** For uniqueness is given in Problem 17.13. □