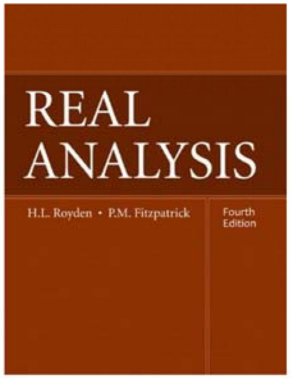
Real Analysis

Chapter 17. General Measure Spaces: Their Properties and Construction

17.3. The Carathédory Measure Induced by an Outer Measure—Proofs



Proposition 17.5

Proposition 17.5. The union of a finite collection of measurable sets is measurable.

Proof. Let E_1 and E_2 be measurable and let A be a subset of X. Then

$$\begin{aligned} \mu^*(A) &= \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c) \text{ since } E_1 \text{ is measurable} \\ &= \mu^*(A \cap E_1) + \mu^*([A \cap E_1^c] \cap E_2) + \mu^*([A \cap E_1^c] \cap E_2^c) \\ &\quad \text{since } E_2 \text{ is measurable and } A \cap E_1^c \subset X. \end{aligned}$$

We have the set identities (1) $(A \cap E_1^c) \cap E_2^c = A \cap (E_1 \cup E_2)^c$ (by DeMorgan's Law), and (2) $(A \cap E_1) \cup (A \cap E_2 \cap E_1^c) = A \cap (E_1 \cup E_2)$ (consider the Venn diagram). So

$$\begin{aligned} \mu^*(A) &= \mu^*(A \cap E_1) + \mu^*([A \cap E_1^c] \cap E_2) + \mu^*([A \cap E_1^c] \cap E_2^c) \text{ by above} \\ &= \mu^*(A \cap E_1) + \mu^*([A \cap E_1^c] \cap E_2) + \mu^*(A \cap [E_1 \cup E_2]^c) \text{ by the first identity} \\ &\geq \mu^*(A \cap [E_1 \cup E_2]) + \mu^*(A \cap [E_1 \cup E_2]^c) \\ &\quad \text{by finite monotonicity and the second identity.} \end{aligned}$$

Proposition 17.5 (continued)

Proposition 17.5. The union of a finite collection of measurable sets is measurable.

Proof (continued).

$$\mu^*(A) \geq \mu^*(A \cap [E_1 \cup E_2]) + \mu^*(A \cap [E_1 \cup E_2]^c).$$

Monotonicity then implies that $E_1 \cup E_2$ is measurable. The general result then follows by induction. □

Proposition 17.6

Proposition 17.6. Let $A \subset X$ and $\{E_k\}_{k=1}^n$ be a finite disjoint collection of measurable sets. Then

$$\mu^*\left(A \cap [\cup_{k=1}^n E_k]\right) = \sum_{k=1}^n \mu^*(A \cap E_k).$$

That is, μ^* is finite additive on the measurable sets (which follows with $A = X$).

Proof. Let $E_1$and E_2 be measurable and disjoint. Then $A \cap [E_1 \dot\cup E_2] \cap E_2^c = A \cap E_1$.

$$\begin{aligned} \mu^*(A \cap (E_1 \dot\cup E_2)) &= \mu^*([A \cap (E_1 \dot\cup E_2)] \cap E_2) + \mu^*([A \cap (E_1 \dot\cup E_2)] \cap E_2^c) \\ &\quad \text{since } E_2 \text{ is measurable and } A \cap (E_1 \dot\cup E_2) \subset X \\ &= \mu^*(A \cap E_2) + \mu^*(A \cap E_1) \text{ by the set identity.} \end{aligned}$$

So the result holds for $n = 2$. By induction, the general result follows. □

Proposition 17.7

Proposition 17.7. The union of a countable collection of measurable sets is measurable.

Proof. Let $E = \cup_{k=1}^{\infty} E_k$ where each E_k is measurable. We may assume without loss of generality that the E_k are disjoint (or else we can replace E_k with $E_k \setminus \cup_{i=1}^{k-1} E_i$ since the measurable sets form an algebra). Let $A \subset X$. For $n \in \mathbb{N}$, define $F_n = \cup_{k=1}^{n} E_k$. Since F_n is measurable and $F_n^c \supset E^c = (\cup_{k=1}^{\infty} E_k)^c$, then

$$\begin{aligned} \mu^*(A) &= \mu^*(A \cap F_n) + \mu^*(A \cap F_n^c) \text{ since } F_n \text{ is measurable} \\ &\geq \mu^*(A \cap F_n) + \mu^*(A \cap E^c) \text{ by monotonicity} \\ &= \sum_{k=1}^{n} \mu^*(A \cap E_k) + \mu^*(A \cap E^c) \text{ by Proposition 17.6} \end{aligned}$$

for all $n \in \mathbb{N}$.

Proposition 17.7 (continued)

Proposition 17.7. The union of a countable collection of measurable sets is measurable.

Proof (continued). Therefore,

$$\begin{aligned} \mu^*(A) &\geq \sum_{k=1}^{\infty} \mu^*(A \cap E_k) + \mu^*(A \cap E^c) \\ &\geq \mu^*(\cup_{k=1}^{\infty}(A \cap E_k)) + \mu^*(A \cap E^c) \text{ by countable monotonicity} \\ &= \mu^*(A \cap (\cup_{k=1}^{\infty} E_k)) + \mu^*(A \cap E^c) \\ &= \mu^*(A \cap E) + \mu^*(A \cap E^c). \end{aligned}$$

So $E = \cup_{k=1}^{\infty} E_k$ is measurable. □

Theorem 17.8

Theorem 17.8. Let μ^* be an outer measure on 2^X. Then the collection $\mathcal{M}$ of sets that are measurable with respect to μ^* is a σ-algebra. If $\overline{\mu}$ is the restriction of μ^* to $\mathcal{M}$, then $(X, \mathcal{M}, \overline{\mu})$ is a complete measure space.

Proof. We've already commented that $\mathcal{M}$ is a σ-algebra. To show $\overline{\mu}$ is a measure space, we must show that $\overline{\mu}(\varnothing) = 0$ (which follows from the definition of outer measure) and that $\overline{\mu}$ is countably additive. For "complete" we need to show that all subsets of measure zero sets are measurable. Let $E_0 \in \mathcal{M}$ where $\mu^*(E_0) = 0$ and $E \subset E_0$. Then, by monotonicity, for all $A \subset \mathbb{R}$ $\mu^*(A \cap E) \leq \mu^*(A \cap E_0) \leq \mu^*(E_0) = 0$, so that $\mu^*(A \cap E) = 0$ for all $A \subset \mathbb{R}$. Also, by monotonicity, $\mu^*(A \cap E^c) \leq \mu^*(A)$ for all $A \subset \mathbb{R}$, so that $\mu^*(A) \geq 0 + \mu^*(A \cap E^c) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$ for all $A \subset \mathbb{R}$. Hence, $E \subset E_0$ is measurable and completeness holds. Monotonicity of μ^* implies subsets of measure zero sets are measure zero.

Theorem 17.8 (continued)

Theorem 17.8. Let μ^* be an outer measure on 2^X. Then the collection $\mathcal{M}$ of sets that are measurable with respect to μ^* is a σ-algebra. If $\overline{\mu}$ is the restriction of μ^* to $\mathcal{M}$, then $(X, \mathcal{M}, \overline{\mu})$ is a complete measure space.

Proof (continued). For countable additivity, suppose $\{E_k\}_{k=1}^{\infty}$ is a sequence of disjoint, measurable sets. Proposition 17.6 gives finite additivity, so we have

$$\begin{aligned} \mu^*\left(\cup_{k=1}^{\infty} E_k\right) &\geq \mu^*\left(\cup_{k=1}^{n} E_k\right) \text{ by monotonicity} \\ &= \sum_{k=1}^{n} \mu^*(E_k) \end{aligned}$$

for all $n \in \mathbb{N}$. Since n is arbitrary, we have $\mu^*\left(\cup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} \mu^*(E_k)$. Countable additivity follows by countable monotonicity. □