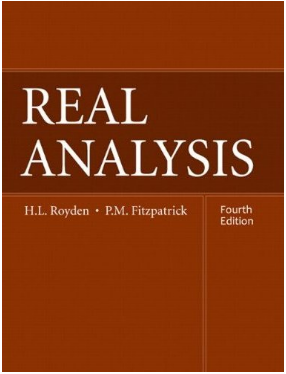
Real Analysis

Chapter 17. General Measure Spaces: Their Properties and Construction

17.3. The Carathéodory Measure Induced by an Outer Measure—Proofs



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Proof. Let E_1 and E_2 be measurable and let A be a subset of X. Then

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We have the set identities (1) $(A \cap E_1^c) \cap E_2^c = A \cap (E_1 \cup E_2)^c$ (by DeMorgan's Law), and (2) $(A \cap E_1) \cup (A \cap E_2 \cap E_1^c) = A \cap (E_1 \cup E_2)$ (consider the Venn diagram).

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$$\begin{aligned} \mu^*(A) &= \mu^*(A \cap E_1) + \mu^*([A \cap E_1^c] \cap E_2) + \mu^*([A \cap E_1^c] \cap E_2^c) \text{ by above} \\ &= \mu^*(A \cap E_1) + \mu^*([A \cap E_1^c] \cap E_2) + \mu^*(A \cap [E_1 \cup E_2]^c) \text{ by the first identity} \\ &\geq \mu^*(A \cap [E_1 \cup E_2]) + \mu^*(A \cap [E_1 \cup E_2]^c) \\ &\quad \text{by finite monotonicity and the second identity.} \end{aligned}$$

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Proof (continued).

$$\mu^*(A) \geq \mu^*(A \cap [E_1 \cup E_2]) + \mu^*(A \cap [E_1 \cup E_2]^c).$$

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Proposition 17.6

Proposition 17.6. Let $A \subset X$ and $\{E_k\}_{k=1}^n$ be a finite disjoint collection of measurable sets. Then

$$\mu^*\left(A \cap [\dot\cup_{k=1}^n E_k]\right) = \sum_{k=1}^n \mu^*(A \cap E_k).$$

That is, μ^* is finite additive on the measurable sets (which follows with $A = X$).

Proof. Let $E_1$and E_2 be measurable and disjoint. Then $A \cap [E_1 \dot\cup E_2] \cap E_2^c = A \cap E_1$.

$$\begin{aligned} \mu^*(A \cap (E_1 \dot\cup E_2)) &= \mu^*([A \cap (E_1 \dot\cup E_2)] \cap E_2) + \mu^*([A \cap (E_1 \dot\cup E_2)] \cap E_2^c) \\ &\quad \text{since } E_2 \text{ is measurable and } A \cap (E_1 \dot\cup E_2) \subset X \\ &= \mu^*(A \cap E_2) + \mu^*(A \cap E_1) \text{ by the set identity.} \end{aligned}$$

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So the result holds for $n = 2$. By induction, the general result follows. □

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Proof. Let $E = \cup_{k=1}^{\infty} E_k$ where each E_k is measurable. We may assume without loss of generality that the E_k are disjoint (or else we can replace E_k with $E_k \setminus \cup_{i=1}^{k-1} E_i$ since the measurable sets form an algebra). Let $A \subset X$.

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$$\begin{aligned} \mu^*(A) &= \mu^*(A \cap F_n) + \mu^*(A \cap F_n^c) \text{ since } F_n \text{ is measurable} \\ &\geq \mu^*(A \cap F_n) + \mu^*(A \cap E^c) \text{ by monotonicity} \\ &= \sum_{k=1}^{n} \mu^*(A \cap E_k) + \mu^*(A \cap E^c) \text{ by Proposition 17.6} \end{aligned}$$

for all $n \in \mathbb{N}$.

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Proof (continued). Therefore,

$$\begin{aligned}
\mu^*(A) &\geq \sum_{k=1}^{\infty} \mu^*(A \cap E_k) + \mu^*(A \cap E^c) \\
&\geq \mu^*(\cup_{k=1}^{\infty}(A \cap E_k)) + \mu^*(A \cap E^c) \text{ by countable monotonicity} \\
&= \mu^*(A \cap (\cup_{k=1}^{\infty} E_k)) + \mu^*(A \cap E^c) \\
&= \mu^*(A \cap E) + \mu^*(A \cap E^c).
\end{aligned}$$

So $E = \cup_{k=1}^{\infty} E_k$ is measurable. □

Theorem 17.8

Theorem 17.8. Let μ^* be an outer measure on 2^X. Then the collection $\mathcal{M}$ of sets that are measurable with respect to μ^* is a σ-algebra. If $\overline{\mu}$ is the restriction of μ^* to $\mathcal{M}$, then $(X, \mathcal{M}, \overline{\mu})$ is a complete measure space.

Proof. We've already commented that $\mathcal{M}$ is a σ-algebra. To show $\overline{\mu}$ is a measure space, we must show that $\overline{\mu}(\varnothing) = 0$ (which follows from the definition of outer measure) and that $\overline{\mu}$ is countably additive.

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Proof (continued). For countable additivity, suppose $\{E_k\}_{k=1}^{\infty}$ is a sequence of disjoint, measurable sets. Proposition 17.6 gives finite additivity, so we have

$$\begin{aligned}\mu^*\left(\cup_{k=1}^{\infty} E_k\right) &\geq \mu^*\left(\cup_{k=1}^{n} E_k\right) \text{ by monotonicity} \\ &= \sum_{k=1}^{n} \mu^*(E_k)\end{aligned}$$

for all $n \in \mathbb{N}$. Since n is arbitrary, we have $\mu^*\left(\cup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} \mu^*(E_k)$. Countable additivity follows by countable monotonicity. □