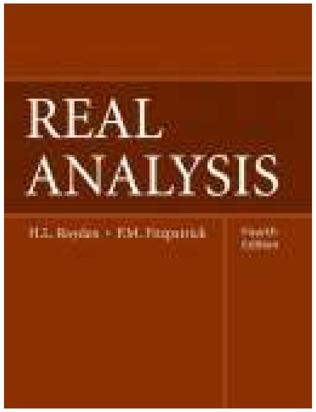
Real Analysis

Chapter 17. General Measure Spaces: Their Properties and Construction

17.4. The Construction of Outer Measure—Proofs



Theorem 17.9

Theorem 17.9. Let $\mathcal{S}$ be a collection of subsets of a set X and $\mu : \mathcal{S} \to [0, \infty]$ a set function. Define $\mu^*(\varnothing) = 0$ and for $E \subset X$, $E \neq \varnothing$, define $\mu^*(E) = \inf\left(\sum_{k=1}^\infty \mu(E_k)\right)$, where the infimum is taken over all countable collections $\{E_k\}_{k=1}^\infty$ of sets in $\mathcal{S}$ that cover E. Then the set function $\mu^* : 2^X \to [0, \infty]$ is an outer measure (called the *outer measure induced* by μ).

Proof. We need only show that μ^* is countably monotone. Let $\{E_k\}_{k=1}^\infty$ be a collection of subsets of X that covers a set E. Without loss of generality, $\mu^*(E_k) < \infty$ for all $k \in \mathbb{N}$, otherwise countable monotonicity follows trivially.

Let $\varepsilon > 0$. For each $k \in \mathbb{N}$, there is a countable collection $\{E_{i,k}\}_{i=1}^\infty$ of sets in $\mathcal{S}$ that covers E_k such that $\sum_{i=1}^\infty \mu(E_{i,k}) < \mu^*(E_k) + \varepsilon/2^k$, by the definition of infimum. Then $\{E_{i,k}\}_{i,k=1}^\infty$ is a sequence of sets in $\mathcal{S}$ which cover $\cup_{k=1}^\infty E_k$ and, therefore, also covers E.

Proposition 17.9 (continued)

Proof (continued). So

$$\begin{aligned}\mu^*(E) &\leq \sum_{i,k=1}^\infty \mu(E_{i,k}) \text{ since } \{E_{i,k}\} \text{ is some specific cover} \\ &= \sum_{k=1}^\infty \left(\sum_{i=1}^\infty \mu(E_{i,k})\right) \\ &\leq \sum_{k=1}^\infty (\mu^*(E_k) + \varepsilon/2^k) \text{ by the above inequality} \\ &= \sum_{k=1}^\infty \mu^*(E_k) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\mu^*(E) \leq \sum_{k=1}^\infty \mu^*(E_k)$ and μ^* is countable monotone. □

Proposition 17.10

Proposition 17.10. Let $\mu : \mathcal{S} \to [0, \infty]$ be a set function defined on a collection $\mathcal{S}$ of subsets of a set X and let $\overline{\mu} : \mathcal{M} \to [0, \infty]$ be the Carathéodory measure induced by μ. Let $E \subset X$ satisfy $\mu(E) < \infty$. Then there is $A \subset X$ for which $A \in \mathcal{S}_{\sigma\delta}$, $E \subset A$, and $\mu^*(E) = \mu^*(A)$. Furthermore, if $E \in \mathcal{M}$ and $\mathcal{S} \subset \mathcal{M}$, then $A \in \mathcal{M}$ and $\overline{\mu}(A \setminus E) = 0$.

Proof. Let $\varepsilon > 0$. Since $\mu^*(E) < \infty$, there is a cover of E by a collection $\{E_k\}_{k=1}^\infty$ of sets in $\mathcal{S}$ for which $\sum_{k=1}^\infty \mu(E_k) < \mu^*(E) + \varepsilon$ by the definition of infimum. Define $A_\varepsilon = \cup_{k=1}^\infty E_k$. Then $A_\varepsilon \in \mathcal{S}_\sigma$ and $E \subset A_\varepsilon$. Since $\{E_k\}_{k=1}^\infty$ is a specific cover of A_ε by elements of $\mathcal{S}$, then

$$\mu^*(A_\varepsilon) \leq \sum_{k=1}^\infty \mu(E_k) \leq \mu^*(E) + \varepsilon.$$

For $k \in \mathbb{N}$, take $\varepsilon = 1/k$ and define $A = \cap_{k=1}^\infty A_{1/k}$. Then A belongs to $\mathcal{S}_{\sigma\delta}$ and $E \subset A$ (since $E \subset A_{1/k}$ for $k \in \mathbb{N}$).

Proposition 17.10 (continued)

Proof (continued). Then

$$\begin{aligned}\mu^*(E) &\le \mu^*(A) \text{ by monotonicity of } \mu^* \\ &\le \mu^*(A_{1/k}) \text{ by monotonicity of } \mu^* \\ &\le \mu^*(E) + 1/k \text{ by above.}\end{aligned}$$

So $\mu^*(E) = \mu^*(A)$.
Now assume that E and each set in $\mathcal{S}$ is μ^* measurable. Since the measurable sets form a σ-algebra, then set A defined above is measurable. The excision property holds for $\overline{\mu}$ since it is a measure (and we have finite additivity), so

$$\begin{aligned}\overline{\mu}(A \setminus E) &= \overline{\mu}(A) - \overline{\mu}(E) \\ &= \mu^*(A) - \mu^*(E) \text{ since } \mu^* \text{ is an extension of } \overline{\mu} \\ &= 0 \text{ because } \mu^*(A) = \mu^*(E) \text{ as shown above.}\end{aligned}$$

□